Set Theory

Lecture 1

introduction; history; motivation

Some historical background

The infinite ∞

Modern **abstract set theory** was first developed by Georg Cantor (1845 – 1918) starting around 1873 as the study of "the infinite."

- The study of mathematical properties of *trans-finite* numbers.
- The theory was only very slowly expanding to an independent branch of mathematics because of the contentious nature of subject matter, namely infinity.

The beginning...

> *Since the science of nature is concerned with magnitudes, change, and time, each of which must be either infinite or finite ... it will be fitting for the student of nature to consider the infinite, [and ask] whether it is or not, and if it is, what it is. (Physics iii.6)*

Aristotelian account

Potential vs. actual infinities

- "The characteristic of the infinite is the opposite of what it is usually said to be. It is not that which has nothing outside it that is infinite, but that which always has something outside it. ... Thus something is infinite if, taking it quantity by quantity, we can always take something outside. On the other hand, that which has nothing outside it is complete and whole" (*Physics* iii.6)
  - Account of what sort of thing thing can be infinite.
  - Arguments for potential but not actual infinity.
- "Annihilation of number" (*Metaphysics* xi.10)
  - If the infinite existed then it would absorb the finite and destroy it.
  - For any finite number a, $a + \infty = \infty$.

Some historical background

The infinite

- Leibniz (ambiguous): He sometimes rejects actual infinity. But ... "I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author."
- Gauss (negative): "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction."
- Bolzano (positive): "A multitude which is larger than any finite multitude, i.e., a multitude with the property that every finite set [of members of the kind in question] is only a part of it, I will call an infinite multitude."

Some historical background

Cantor praises Bolzano for embracing actual infinity but also criticizes him for not having a mathematically clear concept of actual infinity.

- This leads Cantor to develop his own theory of the infinite.

Reception of Cantorian Theory

- Kroneckers: "I don't know what predominates in Cantor's theory – philosophy or theology, but I am sure that there is no mathematics there."
  "God made the integers, all else is the work of man."
- Hilbert: "No one will drive us from the paradise which Cantor created for us."
  After the First International Congress of Mathematics in 1897, Hilbert listed Cantor's **continuum hypothesis** as the No.1 problem in his list of 23 unsolved problem.

What is a set?

Cantor's "definition"

"A set is a collection into a whole of definite distinct objects of our intuition or of our thought. The objects are called the elements (members) of the set."

Example

1. X_1 ={pigs, pigeons, cows, dinosaurs, dragons}
2. $X_2 = \{0, 1, 2, 3\}$
3. $X_3 = \{x \mid x$ is prime number greater than $2\}$
4. $X_4 = \{x \mid x$ is an odd number$\}$
5. $X_5 = \{X_1, X_2, X_3, X_4\}$

Rudimentary properties of Sets

- Their "sizes."
- Their interrelationships: membership $\in$, inclusion $\subseteq$, etc.

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Natural questions

- How big can a set be?
- Is there a set of all sets?

Crisis brewing...

Powersets

Let X be a set, then the **powerset** of X, denoted by $\mathcal{P}(X)$, is the set $\{x \mid x \subseteq X\}$.

Theorem (Cantor)

Every set is strictly "smaller" than its powerset.

- Cantor's diagonalization argument.
- The continuum hypothesis.

Cantor's paradox (1899)

Let U be the set of all sets, then, by Cantor's theorem, its powerset $\mathcal{P}(U)$ is strictly bigger than U. Therefore, U cannot be the set of all sets.

- The set of all sets does not exist.

Logical antinomies

Burali-Forti's paradox (1897)

The set of all ordinal numbers does not exist.

- Cantor discussed this paradox as early as in 1895 and communicated to Hilbert in 1896.
- The mathematicians know that the "naive" theory of sets is in need of revision.

Russell's paradox (1903)

The set of all sets that are *not* members of themselves does not exist.

$$r = \{x \mid x \notin x\}$$

- Russell discovered his paradox through the readings of Cantor's work.
- This paradox is however far more general and worrisome than a particular mathematical one.

The significance of Russell's paradox

Unrestricted Axiom of Comprehension

Let $\varphi(x)$ be any first-order formula with one free variable, then the set $\{x \mid \varphi(x)\}$ exists.

- Intuitively, $\varphi(x)$ is taken to be certain properties, and $\{x \mid \varphi(x)\}$ is the collection of individuals that possess that property.
- To get a contradiction, let $\phi(x) = x \in x$, and let $r = \{x \mid \neg\phi(x)\}$. But r does NOT exist.
- This basic axiom of naive set theory is **false**!

Theorem (in pure logic)

$\neg\exists y \forall x [xRy \leftrightarrow \neg xRx]$

- Russell's paradox $r \in r \leftrightarrow r \notin r$ is a violation of this basic property in pure logic.

The significance of Russell's paradox

Definition (predicativity)

A concept/property is said to be **predicative** if it quantifies over a domain that contains itself, it is **impredicative** otherwise.

- '. . . is yellow,' '. . . is abstract,' '. . . is English.'

? What about the property of being impredicative? Is it predicative or impredicative??

- Russell's paradox highlights problematic nature of impredicative predicates in pure logic.

Self-reference

Broadly speaking, the feature that the crucial entity in question is defined in terms of a totality to which itself belongs.

- The tallest person in the room (Ramsey).
- The set of all sets.

Foundational crises in mathematics

- In the 19th century, mathematicians realized that the "newly" invented calculus, although powerful and widely applicable, is based on a rather shaky foundation (this was known as the second crisis in the foundations of mathematics).
  - Through the efforts of Cauchy, Weierstrass and other, calculus is provided with a rigorous definition based on the ideas of "approximation" and "goes to the limit." (cf. NOTES §1.)
  - Poncaré announced in the 2nd ICM in Paris: "Today there remain in analysis only integers and finite or infinite systems of integers . . . Mathematics . . . has been arithmetized . . . We may say today that absolute rigor has been obtained."
- Ironically enough, at the same time Poncaré made this claim, the third foundational crisis in mathematics was unfolding.

Goals

Develop a rigorous mathematical theory of the infinite.

- finite, countably infinite, higher infinite, well-ordering, ordinal numbers, cardinality, the axiom of choice, etc.

Provide a firm foundation for modern mathematics.

- In light of the paradoxes, set theory needs to be re-evaluated, both for its the basic content and methodologies.
- The axiomatic foundations
- Most mathematics can be "reduced" to this theory.