Sets, Relations & Probability

Lecture 4

Probability

Beginnings

It is difficult to say when and where the measurement of probability started, but there is evidence to show some preliminary forms of it was used in *gambling*.

- Italian mathematician Gerolamo Cardano (1501-76) sometimes makes a living on gambling.
- Galileo advises his patron (the Duke of Tuscan) on how to identify the possible outcomes of dice-rollings

Pascal and Fermat (1654)

The first serious work in the calculus of probability appears to be the correspondence between Pascal and Fermat.

- Interrupted games.
- What is a *fair* price a play gets paid or charged is certain gambling game is interrupted.

Probability

Example (Fair Reward)

A player has undertaken to throw a 6 in 8 throws of a fair six-sided dice. The reward is £6 if he manages to do that. He has now thrown 3 times but without obtaining a 6.

Question: What proportion of the stakes (i.e., £6) would be fair to give the player if he agrees to forgo his fourth throw (just the fourth)?

- This problem was communicated to Pascal by his gambling friend Chevalier de Méré.
- The problem is formulated in terms of *fairness*, but it involves an analysis of random events.

What is probability? (v.1)

Probability is part of mathematics used to model random phenomena.

- Set up a mathematical framework within which random events can be analyzed rigorously.

Basic Concepts of Probability

Definition

The collection of all possible outcomes of an experiment is called the **sample space** of the experiment. An **event** is a well-defined set of possible outcomes of the experiments.

- The set of sample space is also an event – a sure event.
- The concepts of sample space and events can be represented using set-theoretic notations.

Example (Rolling a Dice)

When a fair dice is rolled, the sample space, denoted by S, can be regarded as containing 6 number each representing a possible side of the dice, in symbols,

$$S = \{1, 2, 3, 4, 5, 6\}.$$

The event A that an even number is obtained can be represented as $A = \{2, 4, 6\}$. The event B that a number that is greater than 1 can be represented by $B = \{2, 3, 4, 5, 6\}$.

Basic Concepts of Probability

Null events

Impossible events in a given experiment are referred to as *null events* and are represented by $\varnothing$.

- It's impossible to obtain a negative number in the example above.

NB. Whether an event is considered possible or not is relative to the purposes as well as the details of the underlying modeling.

Complement

The *complement* of an event A is itself an event, denoted by A^c (or $\overline{A}$) such that A does not occur.

- Set-theoretic notation: $A^c = S - A$.
- For example, when roll a dice, the compliment of the event A that an even number is obtained is the event A^c that an event number is NOT obtained (which is equivalent to the event that an odd number is obtained). We have $A^c = S - A = \{1, 3, 5\}$.

Basic Concepts of Probability

Union

The **union** of event A and event B is an event such that either event A is obtained or event B is obtained (or both).

- Set-theoretic notation: $A \cup B$
- In the running example, the union of the event that an even number is obtained and that an odd number is obtained is a sure event.

Intersection

The **intersection** of event A and event B is an event such that both event A and event B are obtained.

- Set-theoretic notation: $A \cap B$
- In the running example, the intersection of the event that an odd number is obtained and the event that a number that is great 1 is obtained is the event that either number 3 or 5 is obtained.

Basic Concepts of Probability

Example (Tossing a Coin)

Suppose that a coin is tossed three times. Set up the sample space

s_1 :	s_5 :
s_2 :	s_6 :
s_3 :	s_7 :
s_4 :	s_8 :

Represent the following four events:

A: at least one head is obtained in the three tosses.
B: a head is obtained on the second toss.
C: a tail is on the third toss.
D: no heads are obtained.

Basic Concepts of Probability

Disjoint events

Two events A and B are said to be **disjoint** if $A \cap B = \varnothing$. Let $\{A_1, A_2, \ldots, A_n\}$ be a collection of events, these events are said to be **mutually disjoint** or **mutually exclusive** if any two distinct events in the collection are disjoint. In symbols, $A_i \cap A_j$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.

Partition of an event

A collection of events $\{A_1, A_2, \ldots, A_n\}$ is said to be a **partition** of an event B if

1. $A_1, A_2, \ldots, A_n$ are mutually exclusive; and
2. $A_1 \cup A_2 \cup \cdots \cup A_n = B$.

Some properties of operations on events

- $(A^c)^c = A \qquad \varnothing^c = S \qquad S^c = \varnothing \qquad A \cup B = B \cup (A \cap B^c)$
- (De Morgan's Law) $(A \cup B)^c = A^c \cap B^c \qquad (A \cap B)^c = A^c \cup B^c$

Probability Calculus

Axiomatic System

We can now give a mathematical definition of probability, which is based on an axiomatic system developed by Andrei Kolmogorov in 1933.

Axiom 1 For every event A, $\Pr(A) \geq 0$.

Axiom 2 $\Pr(S) = 1$.

Axiom 3 For any collection of mutually disjoint events

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \Pr(A_1) + \Pr(A_2) + \cdots + \Pr(A_n).$$

Definition

A **probability measure** or simply **probability** on a given sample space S is a function Pr such that it specifies a number $\Pr(A)$ for every event A and that it satisfies Axioms 1–3.

NB. There are different *interpretations* of the notion of probability, but all interpretations follow the same probability *calculus*.

Probability Calculus

Properties of probability

- $\Pr(\varnothing) = 0$.
- For every event A, $\Pr(A^c) = 1 - \Pr(A)$.
- If $A \subseteq B$, then $\Pr(A) \leq \Pr(B)$.
- For every event A, $0 \leq \Pr(A) \leq 1$.
- For every two events A and B, $\Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B)$.
- For any two events A, B , $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

Example

A patient arrives at a doctor's office with a sore throat and low fever. After an exam, the doctor decides that the patient has either a bacterial infection or a viral infection or both. The doctor decides that there is a probability of 0.7 that the patient has a bacterial infection and a probability of 0.4 that the person has a viral infection. What is the probabilities that the patient has both infections?

Probability Calculus

Sample space and probabilities

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a sample space. A probability Pr on S assigns for each s_i a number p_i ($i = 1, 2, \ldots, n$). By the axioms, these numbers satisfy the following two conditions

1. $p_i \geq 0$ for $i = 1, 2, \ldots, n$; and
2. $p_1 + p_2 + \cdots + p_n = 1$.

Simple Sample Spaces

A sample space S containing n outcomes $s_1, s_2, \ldots, s_n$ is called a **simple sample space** $s_1, s_2, \ldots, s_n$ are considered equally probable, that is, if the probability assigned to each outcomes $s_1, s_2, \ldots, s_n$ is $1/n$. If an event A in this simple sample space contains exactly m outcomes, then

$$\Pr(A) = \frac{m}{n}.$$

NB. This is sometimes referred to as the classical interpretation of probability, which is based on certain *symmetry* considerations.

Probability Calculus

Example (Roll Two Dice)

Consider an experiment in which two balanced dice are rolled, and the sums of the two numbers appearing on each dice are observed (for example, if one dice shows 2 and the other shows 5 then the sum is 7). What is the probability a sum of 9 is observed?

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)