ON UNFORM DISTRIBUTION OVER NATURAL NUMBERS

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A uniformly distributed probabilistic measure on natural numbers $\mathbb{N}$ is of particular interest because (1) it serves a good purpose of delineating the difference between finite additivity and countable additivity; (2) its use is often tied to the notion of randomness: it amounts to saying that choose a number "at random." The latter is commonly understood in the following relative frequentist interpretation of uniformity of natural numbers. Let A be any subset of $\mathbb{N}$. For each number $n < \infty$, denote the number of elements in A that are less or equal to n by $A(n)$, that is,

$$A(n) = \big|A \cap \{1, \ldots, n\}\big|. \tag{1}$$

Define the *density* of A by the limit (if exists)

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}. \tag{2}$$

Let $\mathcal{C}_d$ be the collection of all sets of natural numbers that have densities. The following properties of the density function are easy to verify.

Proposition 1. (1) $d(\varnothing) = 0$ and $d(\mathbb{N}) = 1$.
(2) For each natural number n, $d(\{n\}) = 0$.
(3) For any finite $A \in \mathcal{C}_d$, $d(A) = 0$.
(4) If $A, B, A \cup B \in \mathcal{C}_d$ and $A \cap B = \varnothing$, then $d(A \cup B) = d(A) + d(B)$.
(5) If $A \in \mathcal{C}_d$, then, for any number n, $A + n \in \mathcal{C}_d$ and $d(A) = d(A + n)$, where $A + n = \{x + n \mid x \in A\}$.
(6) The set of even numbers has density $1/2$, or more generally, the set of numbers that are divisible by $m < \infty$ has density $1/m$.

Notice that d is not defined for *all* subsets of $\mathbb{N}$ ($\mathcal{C}_d$ is not a field of natural numbers). We hence seek to extend d to a finitely additive probability measure μ so that μ is defined for all subsets of the natural numbers and that μ agrees with d on $\mathcal{C}_d$ (Theorem 6 below). One version of the extension theorem has been given by Rao and Rao (1983, Theorem 3.2.10).[1] The set-theoretic approach explicated in the next subsection is from Hrbacek and Jech (1999, Ch. 11).

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[1] Kadane and O'Hagan (1995, Theorem 1) show that the monotonicity condition given by Rao and Rao (1983) in their extension theorem is also necessary, see also Schirokauer and Kadane (2007).

Filter and Ultrafilter. A *filter* on a nonempty set S is a collection $\mathcal{F}$ of subsets of S such that

(1) $S \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$,
(2) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$,
(3) if $X, Y \subseteq S$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.

Example 2. (1) A trivial filter $\mathcal{F} = \{S\}$.
(2) Let $A \subseteq S$, a *principal filter* generate by A is the collection $\{X \subseteq S | A \subseteq X\}$. In the case of natural numbers where $S = \mathbb{N}$, a principal filter generated by $n_0 < \infty$ is the collection $\mathcal{F}_{n_0}$ of sets of numbers such that $X \in \mathcal{F}_{n_0}$ if and only if $n_0 \in X$.
(3) As for an example of a *nonprinciple filter*, let S an infinite set, the *Fréchet filter* on S is the collection

$$\mathcal{F} = \{X \subseteq S \mid S - X \text{ is finite}\}. \tag{3}$$

That is, $\mathcal{F}$ is the filter of all *cofinite* subsets of S. ◁

A filter $\mathcal{U}$ is said to be an *ultrafilter* if, for each $X \subseteq S$, either $X \in \mathcal{U}$ or $S - X \in \mathcal{U}$. The following extension theorem (due to Tarski, 1930) is crucial to our construction of an finitely additive probability measure on $\mathcal{P}(\mathbb{N})$. The proof uses Zorn's lemma and is widely available (see, for instance, Jech, 2003, §7).

Theorem 3 (Tarski)**.** Every filter can be extended to an ultrafilter.

Recall that our main concern in the last subsection is that the density function $d(\cdot)$ is not defined for all the subset of natural numbers, in other words, there exists some $A \subseteq \mathbb{N}$ such that the sequence $\{A(n)/n\}_{n=1}^{\infty}$ does not converge. The goal is to extend d to some measure so that (2) holds for all A's. To this end, we define a general notion of *convergence in a ultrafilter*, which has the property that, given an ultrafilter of natural numbers, *every* bounded sequences converges. As we shall see, this leads to the extension of d to $\mathcal{P}(\mathbb{N})$ as required.

Definition 4. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. For some $a \in \mathbb{R}$, $\{a_n\}_{n=1}^{\infty}$ is said to be *convergent in* $\mathcal{U}$ to a (or a is a $\mathcal{U}$*-limit* of the sequence), written $a = \lim_{\mathcal{U}} a_n$, if for every small $\epsilon > 0$,

$$\{n \mid |a_n - a| < \epsilon\} \in \mathcal{U}. \tag{4}$$

Lemma 5. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, then, for any bounded real sequence $\{a_n\}$, there exists a unique $\mathcal{U}$-limit.

Proof. Since $\{a_n\}$ is bounded, for every $x < \infty$, let

$$A_x = \{n \mid a_n < x\}.$$

Further, let

$$a = \sup\{x \mid A_x \notin \mathcal{U}\}.$$

We show that $\lim_{\mathcal{U}} a_n = a$, that is, we show that, for any $\epsilon > 0$, (4) holds. Note that, for any $x < y$, $A_x \subseteq A_y$, hence if $A_x \in \mathcal{U}$ then $A_y \in \mathcal{U}$. Since a is the least upper bound of x for which $A_x \notin \mathcal{U}$, we have $A_{a+\epsilon} \in \mathcal{U}$ but $A_{a-\epsilon/2} \notin \mathcal{U}$. Given that $\mathcal{U}$ is an ultrafilter, the latter implies that $S - A_{a-\epsilon/2} \in \mathcal{U}$, that is,

$$S - A_{a-2\epsilon} = \left\{ n \;\middle|\; a - \frac{\epsilon}{2} \leq a_n \right\} \in \mathcal{U}.$$

Since $A_{a+\epsilon} = \{n \mid a_n < a + \epsilon\} \in \mathcal{U}$ and $\{n \mid a - \epsilon/2 \leq a_n\} \subseteq \{n \mid a - \epsilon < a_n\}$, we have that $\{n \,\big|\, |a_n - a| < \epsilon\} = \{n \mid a_n < a + \epsilon\} \cap \{n \mid a - \epsilon < a_n\} \in \mathcal{U}$, and hence (4). To show uniqueness, note that if there is some $b \neq a$ such that $b = \lim_{\mathcal{U}} a_n$. Let $\epsilon = |a - b|$, then, by (4), both $A = \{n \mid |a_n - a| < \epsilon/2\}$ and $B = \{n \mid |a_n - b| < \epsilon/2\}$ are in $\mathcal{U}$. Clearly, $A \cap B = \varnothing$, and hence $B \subseteq S - A$. But this implies, from $B \in \mathcal{U}$ and the fact that $\mathcal{U}$ is a ultrafilter, that $S - A$ is also in $\mathcal{U}$, which is impossible. □

Theorem 6. There exists a finitely additive probability measure on all subsets of $\mathbb{N}$ that extends the density function d.

Proof. Let $\mathcal{U}$ be a Fréchet ultrafilter on $\mathbb{N}$ (the existence of $\mathcal{U}$ is guaranteed by Example 2 (3) and Theorem 3). Define a measure μ on $\mathcal{P}(\mathbb{N})$ to be such that

$$\mu(A) = \lim_{\mathcal{U}} \frac{A(n)}{n}, \tag{5}$$

where $A(n)$ is defined as in (1). By Lemma 5, μ is well defined for all $A \in \mathcal{P}(\mathbb{N})$. Note that, for any A, if $d(A)$ exists, say $d(A) = a$, then $a = \mu(A)$. For, by definition, if for any small ϵ there exists some N such that, for all $n > N$, $|A(n)/n - a| < \epsilon$, then, given that $\mathcal{U}$ is the ultrafilter of all cofinite subsets of $\mathbb{N}$, it follows that $\{n \,\big|\, |A(n)/n - a| < \epsilon\} \in \mathcal{U}$, and hence $\mu(A) = a$.

It remains to show that μ is indeed a finitely additive probability measure. Clearly, $\mu(\varnothing) = 0$ and $\mu(\mathbb{N}) = 1$. We show μ is finitely additive. To this end, let A, B be any disjoint subsets of $\mathbb{N}$. By (5) and the fact that $A \cap B = \varnothing$,

$$\begin{aligned}
\mu(A \cup B) &= \lim_{\mathcal{U}} \frac{(A \cup B)(n)}{n} \\
&= \lim_{\mathcal{U}} \frac{A(n) + B(n)}{n} \\
&= \lim_{\mathcal{U}} \frac{A(n)}{n} + \lim_{\mathcal{U}} \frac{B(n)}{n} = \mu(A) + \mu(B).
\end{aligned}$$

(Actually, it can also be easily seen that μ is also translation-invariant.) Therefore, μ is a measure defined for all subsets of $\mathbb{N}$ that extends the density function d. □

The following is a classical example of finitely but not countably additive probability measure on the natural numbers which is a simple form of the density function d introduced above.

Example 7. Let $\{\lambda_n\}$ be a sequence of functions defined on $\mathbb{N}$ such that

$$\lambda_n(i) = \begin{cases} 1/n & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n. \end{cases} \tag{6}$$

Clearly, each $\lambda_n(i)$ takes the form of $A(n)/n$ in (2) where $A = \{i\}$, and $\{\lambda_n\}$ converges point-wisely to the density function d (on singletons). By Theorem 6, there exists a function λ defined for all subsets of $\mathbb{N}$ that extends d. Further, by Proposition 1, λ satisfies the following properties:

(1) λ is defined for all subsets of $\mathbb{N}$.
(2) $\lambda(\varnothing) = 0$ and $\lambda(\mathbb{N}) = 1$.
(3) λ is finitely additive.
(4) λ is *not* countably additive.
(5) For any $i < \infty$, $\lambda(i) = 0$. (Again, we write $\lambda(i)$ for $\lambda(\{i\})$).
(6) For any $A \subseteq \mathbb{N}$, if A is finite then $\lambda(A) = 0$; if A is cofinite (i.e. if $\mathbb{N} - A$ if finite) then $\lambda(A) = 1$.
(7) $\lambda(\{2n \mid n \in \mathbb{N}\}) = 1/2$, i.e., the set of even numbers has measure $1/2$; or, in general, the set of numbers that are divisible by $m < \infty$ has measure $1/m$. As a result of this property, we have that the assignment of μ can be arbitrarily small: for any $\epsilon > 0$, there exists some n such that the set of numbers that are divisible by n has measure $1/n < \epsilon$. $\triangleleft$

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