Set Theory

Lecture 8

axiom of choice, cardinal arithmetic

Recap

Ordinal and cardinal numbers

- Each well-ordered set can be identified with a unique **ordinal number**.
- Cardinal numbers are "representatives" among (well-ordered) sets with the same cardinalities.
- An ordinal number α is a **cardinal number** if $|\beta| \neq |\alpha|$ for all $\beta < \alpha$.
  - α is also called an initial ordinal with the property that α is not equinumerous to any $\beta < \alpha$.
  - Using cardinal numbers, we can characterize the cardinality of any (well-ordered) sets.

Definition (cardinality)

Let W be any <u>well-ordered set</u>, the **cardinality** of W, denoted by $|W|$, is the least ordinal α such that $|W| = |\alpha|$.

Question: Which sets can be well-ordered?

Well-orderable Sets

Cantor's Well-ordering Principle: Every set X can be well ordered by some binary relation $(\preceq)$ defined on X.

- "A law of thought which appears to me to be fundamental, rich in its consequences, and particularly remarkable for its general validity." (Cantor 1883)

"Proof".

Given a set X, let a be an element that is not in X. Define a one-to-one function from some ordinal λ onto X as follows

$$f(0) = \begin{cases} \text{some element of } X & \text{if } X \neq \varnothing, \\ a & \text{otherwise.} \end{cases}$$

$$f(1) = \begin{cases} \text{some element of } X - \{f(0)\} & \text{if } X - \{f(0)\} \neq \varnothing, \\ a & \text{otherwise.} \end{cases}$$

Well-orderable Sets

"Proof" (continued).

In general,

$$f(\alpha) = \begin{cases} \text{some element of } X - \operatorname{ran}(f \restriction \alpha) & \text{if } X - \operatorname{ran}(f \restriction \alpha) \neq \varnothing, \\ a & \text{otherwise.} \end{cases}$$

Let λ be the least ordinal such that $f(\lambda) = a$, then we establish an well-ordering of X (via f and ordinal λ).

- There is however a glitch in this "proof": how do we know this function f exists?

Recall that to justify f we use transfinite recursion, but this means that there has to be an operation G already defined on X s.t.

$$\begin{aligned} &G(f \restriction \alpha) \in X - \operatorname{ran}(f \restriction \alpha) && \text{if } X - \operatorname{ran}(f \restriction \alpha) \neq \varnothing, \\ &G(f \restriction \alpha) = a && \text{otherwise.} \end{aligned}$$

Well-orderable Sets

Definition (choice function)

Let S be any system of sets. A function g on S is called a **choice function** for S if $g(Y) \in Y$ for any nonempty $Y \in S$.

- To fill the gap in the "proof" above, **assume** that such a choice function g exists for $\mathcal{P}(X)$, **then** the missing operation G can be defined as

$$G(x) = \begin{cases} g(X - \operatorname{ran} x) & \text{if } x \text{ is a function and } X - \operatorname{ran}(x) \neq \varnothing, \\ a & \text{otherwise.} \end{cases}$$

- This leads to the following result by Zermelo (1904/08).

Theorem (Well-ordering)

A set X is well orderable if (and only if) $\mathcal{P}(X)$ has a choice function.

The Axiom of Choice

Axiom of Choice (AC): There exists a choice function for every system of sets.

In symbols: $\forall X \big[\varnothing \notin X \to (\exists f : X \to \bigcup X)(\forall x \in X) f(x) \in x\big]$

Some consequences of the Axiom of Choice

1. For every infinite set S there exists a unique aleph $\aleph_\alpha$ such that $|S| = \aleph_\alpha$.
2. Every infinite set has a countable subset.
3. If f is a function and A is a set, then $|f[A]| \leq |A|$.
4. The union of a countable collection of countable sets is countable (mentioned in Lecture 5).
5. The set of all reals is not the union of countably many countable sets.
6. The Continuum Hypothesis: $2^{\aleph_0} = \aleph_1$.

The Axiom of Choice

Equivalent formulations of the Axiom of Choice (in ZF)

1. (Well-ordering Theorem) Every set can be well-ordered.
2. (Zorn's Lemma) If every chain in a partially ordered set P has an upper bound then P has a maximal element.

A brief chronology of AC

- **1904/08** Zermelo explicitly formulates AC and uses it to prove the well-ordering theorem.
- **1922** Fraenkel introduces the a method to establish independence of AC from set theory with atoms.
- **1924** Banach and Tarski derive from AC their paradoxical decompositions of the sphere.
- **1935-8** Gödel proves relative consistency of AC with ZF.
- **1963** Cohen proves independence of AC from ZF.

Cardinal Arithmetic

In the presence of AC, every set is well-orderable and has a unique cardinal number as its cardinality. We can define an arithmetic among cardinal numbers.

Definition (addition)

$\kappa + \lambda = |A \cup B|$, where $|A| = \kappa$, $|B| = \lambda$, and $A \cap B = \varnothing$.

Some properties of cardinal addition

1. $\kappa + \lambda = \lambda + \kappa$.
   - Unlike ordinal arithmetic, cardinal addition is commutative.
2. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.
3. $\kappa \leq \kappa + \lambda$.
4. If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1 + \lambda_1 \leq \kappa_2 + \lambda_2$.

N.B. Not all laws of addition of number hold for addition of cardinals.

- As noted above, $\aleph_0 + \aleph_0 = \aleph_0$.

Cardinal Arithmetic

Definition (multiplication)

$\kappa \cdot \lambda = |A \times B|$, where $|A| = \kappa$ and $|B| = \lambda$.

Some properties of cardinal multiplication

1. $\kappa \times \lambda = \lambda \times \kappa$.
   - Unlike ordinals, cardinal multiplication is commutative.
2. $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$
3. $\kappa \cdot (\lambda + \mu) = (\kappa \cdot \lambda) + (\kappa \cdot \mu)$.
   - $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
4. $\kappa \leq \kappa \cdot \lambda$ if $\lambda > 0$.
5. If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1 \cdot \lambda_1 \leq \kappa_2 \cdot \lambda_2$.
6. $\kappa + \kappa = 2 \cdot \kappa$.
   - If $|A| = \kappa$, then $2 \cdot \kappa$ is the cardinal of $\{0, 1\} \times A$.
7. $\kappa + \kappa \leq \kappa \cdot \kappa$, when $\kappa \geq 2$.

Cardinal Arithmetic

Definition (exponentiation)

$\kappa^\lambda = |A^B|$, where $|A| = \kappa$ and $|B| = \lambda$.

- A^B is the set of all functions mapping from B to A.

Some properties of cardinal multiplication

1. $\kappa \leq \kappa^\lambda$ if $\lambda > 0$.
2. $\lambda \leq \kappa^\lambda$ if $\kappa > 1$.
3. If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1^{\lambda_1} \leq \kappa_2^{\lambda_2}$.
4. $\kappa \cdot \kappa = \kappa^2$.
   - To show this, let $|A| = \kappa$, find one-to-one correspondence between $A \times A$ and $A^{\{0,1\}}$ (i.e., the set of all functions mapping from $\{0, 1\}$ into A).

Cardinal Arithmetic

Theorem

1. $\kappa^{\lambda+\mu} = k^\lambda \cdot \kappa^\mu$.
2. $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

Proof.

Continuum Hypothesis Again

Theorem

$|\mathcal{P}(X)| = 2^{|X|}$ *for every set* X*.*

Proof.

The GCH

Cantor's Theorem can now be restated in terms of any cardinals

$$\kappa < 2^\kappa \quad \text{for all cardinal } \kappa.$$

The Generalized Continuum Hypothesis (GCH) says that for any ordinal α,

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$