Set Theory

Lecture 7

axiom of foundation, cardinal numbers

Recap

Ordinal numbers

- A set α is an **ordinal number** if
  i. α is transitive,
  ii. α is well-ordered by $\in$.
- Well-ordered sets and their order types
- Arithmetic of ordinal numbers.
  - Axiom of replacement
  - Transfinite induction and recursion
  - Ordinal addition and multiplication are in general not commutative.

Well-founded sets

- The transitive closure of a given set
- Well-founded relations

Well-founded Sets

Well-founded relations

Let R be a binary relation in A, and $X \subseteq A$. Say that $a \in X$ is an **R-minimal** element of X if there is no $x \in X$ such that xRa.
R is **well-founded** on A if every nonempty subset of A has an R-minimal element.

- Clearly, for any well-ordered set $(\prec, A)$, $\prec$ is a well-founded relation in A.
- The membership relation $\in$ is well-founded in any ordinal numbers.

Definition

A transitive set T is well-founded if and only if the membership relation $\in$ is well-founded on T.

- That is, for every $X \subseteq T$, $X \neq \varnothing$, there is $x \in X$ such that $x \cap X = \varnothing$.

Well-founded Sets

Theorem

*For any set X, there exists a smallest transitive set containing X as a subset, called the **transitive closure** of X, and denoted by*

$$\mathrm{TC}(X).$$

Further, $Y \in \mathrm{TC}(X)$ iff there is a finite sequence $\langle x_0, x_1, \ldots, x_n \rangle$ such that $x_0 = X$, $x_i \in x_{i+1}$, and $x_n = Y$.

Proof.

Well-founded Sets

Definition

A set X is **well-founded** if $\mathrm{TC}(X)$ is a well-founded set.

- If X is well-founded then there is no sequence $\langle X_n \mid n \in \mathbb{N}\rangle$ such that $X_0 = X$ and $X_{n+1} \in X_n$ for all $n \in \mathbb{N}$.
  - Consequently, a well-founded set cannot be an element of itself (i.e., let $X_n = X$ for all $n \in \mathbb{N}$ above).
- The axioms we have so far do not preclude $X \in X$.

N.B. The set of all sets, if exists, is an example of a set that contains itself as an element. The fact that the set of all sets does not exist does not necessarily imply that there is no set that is an element of itself.

In **ZFC**, we choose to work with well-founded sets only.

Axiom of Foundation (Regularity): Every non-empty set X has an $\in$-minimal element.

In symbols: $\forall X[X \neq \varnothing \rightarrow \exists x \in X(x \cap X = \varnothing)]$

Well-founded Sets

The cumulative hierarchy of well-founded sets

$$\begin{aligned} V_0 &= \varnothing; \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \quad \text{for all } \alpha; \\ V_\alpha &= \textstyle\bigcup_{\beta<\alpha} V_\beta \quad \text{for all limit } \alpha. \end{aligned}$$

- $V_1 = \{\varnothing\}$, $V_2 = \{\varnothing, \{\varnothing\}\}$, $V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

Denote the universe of well-founded sets by

$$V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha.$$

Some properties of V

1. If $\beta < \alpha$ then $V_\beta \subseteq V_\alpha$.
2. For all α, V_α is transitive and well-founded.
3. For any α, $\alpha \subset V_\alpha$.

Well-founded Sets

Why (and why not) the axiom of foundation?

- It provide a **cumulative hierarchy**: each "layer" consists of all sets whose members are taken from the lower layers.
  - A set X is well-founded if and only if there is some α such that $X \in V_{\alpha+1}$ [the least α for which $X \in V_{\alpha+1}$ is also called the rank of X, denoted $\mathrm{rank}(X)$].
  - In V, we will never find a set X such that $X \in X$, as all members of a set in a given layer belong to lower layers.
- A set X such that $X \in X$ by itself is not vicious in the way that the set of all set is. The axiom of foundation amounts to a more restrictive definition of "set".
- Unlike other axioms of **ZFC**, the presence or the absence of the axiom of foundation makes no difference as far as the development of ordinary mathematics within set theory is concerned.

Cardinal Numbers

Cardinality

Recall that we have defined a comparative notion of cardinality between any two sets (i.e., $|A| = |B|$ and $|A| \leq |B|$) in terms of equinumerosity. The next goal is to define an "absolute" notion of cardinality.

The Frege-Russell definition

The simplest way is to define the cardinality of X as the set of all sets that are equinumerous to X.

- That is, to use the equivalence class itself as the cardinal number for members of the class.
- But in **ZFC**, such a set does not usually exist:
  - For instance, let W be set of all singletons, then, by this definition, W is the cardinality of any set with one member (i.e., cardinal number 1), but $\bigcup W$ becomes the set of all sets, which however does not exist.

Cardinal Numbers

Informal Characterization

Find a "representative" among (well-orderable) sets with the same cardinalities.

- This is straightforward for finite sets, in which case we use the natural numbers as representatives (finite cardinals).
- For any infinite X, we in general cannot use the ordinal number of the same order type to represent its cardinality.
  - There are many ordinal numbers of the same cardinality.

  e.g. $\omega, \omega+1, \ldots, \omega \cdot \omega, \omega^2+1, \ldots$
- But this is easy to fix: we can simply take the least ordinal number among ordinals of the same cardinality as the representative of that cardinality.
- For any general well-ordered set X, we can also take the least ordinal among sets of the same cardinality as the representative of that cardinality.

N.B. This definition requires the axiom of choice.

Cardinal Numbers

Formal Definitions

Definition (cardinals)

An ordinal number α is a **cardinal number** if $|\beta| \neq |\alpha|$ for all $\beta < \alpha$.

- α above is sometimes called an **initial ordinal**: it is not equinumerous to any $\beta < \alpha$.
- As a notational convention, we use κ, λ, ν to range among cardinal numbers
- Any finite number $n \in \mathbb{N}$ is a cardinal number.
- ω is the first/least infinite cardinal.

Definition (cardinality)

Let W be any well-ordered set, the **cardinality** of X, denoted by $|W|$, is the least ordinal α such that $|W| = |\alpha|$.

- Clearly, $|W|$ is a cardinal.

Cardinal Numbers

Theorem

1. *For any cardinal κ, there is a cardinal number greater than κ.*
2. *If X is a set of cardinals, then $\sup X$ is a cardinal.*

- This allows us to define a "scale" of larger and larger infinite cardinals (by transfinite recursion).

Definition

Let κ be a cardinal, the **successor** of κ, denoted by κ^+, is the least cardinal number greater than κ.

- In light of the theorem above, every cardinal has a successor.
- But not every cardinal is the successor of another cardinal.
  - ω is not the successor of any $n \in \mathbb{N}$.

Cardinal Numbers

A cardinal number is therefore either finite or infinite. As a convention, the infinite cardinals are called **alephs**.

Alephs

We use $\aleph_\alpha$ when referring to cardinal numbers, and ω_α to denote order-type:

$$\begin{aligned} \aleph_0 &= \omega_0 = \omega; \\ \aleph_{\alpha+1} &= \omega_{\alpha+1} = \aleph_\alpha^+; \quad \text{for all ordinal } \alpha \\ \aleph_\alpha &= \sup\{\omega_\beta \mid \beta < \alpha\} \quad \text{for all limit ordinal } \alpha. \end{aligned}$$

- $|\mathbb{N}| = \aleph_0$.
- A cardinal $\aleph_{\alpha+1}$ is a <u>successor cardinal</u>.
- A cardinal whose index is a limit ordinal is a <u>limit cardinal</u>.

The never-ending cardinals

$0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots \aleph_{17}, \ldots \aleph_\omega, \ldots \ldots \aleph_{\omega^{\omega^\omega}}, \ldots \ldots$

Cardinal Numbers

We have now defined a scale of larger and larger cardinals ($\aleph_0 < \aleph_1 < \aleph_2, \ldots$). There are other ways to demonstrate that the cardinalities of sets may differ:

Theorem (Cantor)

$|X| < |\mathcal{P}(X)|$ for every set X.

Proof.

Cardinal Numbers

Theorem

$|\mathcal{P}(X)| = \left|2^X\right|$, for every set X.

Proof.

The Continuum Hypothesis

1. Let X in the Cantor' theorem be $\mathbb{N}$, we have $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.
2. By the last theorem, $|\mathcal{P}(\mathbb{N})| = \left|2^{\mathbb{N}}\right| = 2^{\aleph_0}$.
3. It can be shown that the cardinality of the set of all reals $|\mathbb{R}| = \left|2^{\mathbb{N}}\right| = 2^{\aleph_0}$.

This yields

$$|\mathbb{N}| = \aleph_0 < 2^{\aleph_0} = |\mathbb{R}|.$$

Question: where does $2^{\aleph_0}$ fit on our scale of cardinals?

$$\aleph_0, \aleph_1, \ldots \aleph_{17}, \ldots \aleph_\omega, \ldots\ldots \aleph_{\omega^{\omega^\omega}}, \ldots\ldots$$

The Continuum Hypothesis (CH): There is no uncountable cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$, that is,

$$2^{\aleph_0} = \aleph_1.$$

The Continuum Hypothesis

A brief history of CH

1878 The problem was first raised by Georg Cantor.

1900 David Hilbert ranked CH as the first problem (on a list of 23 unsolved problems) announced during the 2nd International Congress of Mathematics in Paris.

1939 Kurt Gödel showed that CH is consistent with the axioms of set theory (constructible universe).

1963 Paul Cohen proved that $\neg$CH is consistent with the axioms of set theory (forcing).

- Put Gödel's and Cohen's results together, we have that CH is independent of **ZFC**.