Set Theory

Lecture 5

finite, countable, and uncountable sets

Recap

Axiomatic set theory (so far)

- The Axioms of Extensionality, Pairing, Union, Powerset.
- The Axiom Schema of Comprehension (separation).
- The Axiom of Infinity.

The natural numbers

- Define the natural numbers in terms of sets.

$$\begin{aligned} 0 &= \varnothing \\ n+1 &= n \cup \{n\} \\ \mathbb{N} &= \bigcap \{X \mid X \text{ is inductive}\} \end{aligned}$$

- Properties of the natural numbers: the Inductive Principle (two versions)
- The Recursion Theorem – it allows us to define various arithmetic operations.

Cardinality of sets

Definition (equinumerosity)

Sets A and B are said to be **equinumerous** (or **equipotent**) if there is a one-to-one function f mapping from A onto B, denoted

$$|A| = |B|.$$

- f is a bijection (one-to-one correspondence) between A and B.

e.g. $2 = \{\varnothing, \{\varnothing\}\}$ and $\{\{\varnothing\}, \{\{\varnothing\}\}\}$.

Definition

The **cardinality** of A is less or equal to that of B if there is a one-to-one function f mapping from A into B, denoted

$$|A| \leq |B|.$$

Cardinality of sets

Theorem (Cantor-Schröder-Bernstein)

If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.

- Known as the first "non-trivial result" in set theory.

1887 Cantor publishes the theorem, however without proof.

1895 Cantor states the theorem in his first paper on set theory and transfinite numbers. He obtains it as an easy consequence of the linear order of cardinal numbers.

1896 Schröder publishes a proof sketch which, however, is shown to be faulty by Korselt in 1911 (confirmed by Schröder).

1897 Bernstein, a 19 years old student in Cantor's Seminar, presents his proof.

1897 Almost simultaneously, Schröder finds a proof.

1898 Bernstein's proof (not relying on the axiom of choice) is published by mile Borel in his book on functions.

Cardinality of sets

Theorem (Cantor-Schröder-Bernstein)
If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.

Proof.

Finite sets

Finite sets can be defined in terms of the natural numbers.

Definition
A set S is **finite** if it is equinumerous to some natural number $n \in \mathbb{N}$. We say that <u>S has n elements</u>, in symbols

$$|S| = n.$$

A set is **infinite** if it is not finite.

- By our definition, cardinal numbers (will be defined rigorously) of finite sets are the natural numbers.
- Natural numbers themselves are finite sets, we have $|n| = n$ for all $n \in \mathbb{N}$.
- The above definition is well defined, that is, the cardinal number of a finite set is unique.
  - ? Let $n \in \mathbb{N}$, show that there is no one-to-one mapping of n onto a proper subset $X \subseteq n$.

Finite sets

Some properties of finite sets

1. If X is a finite set and $Y \subseteq X$, then Y is finite. Moreover, $|Y| \leq |X|$.
2. If X is a finite set and f is a function on X, then $f[X]$ is finite. Moreover, $|f[X]| \leq |X|$.
3. If X and Y are finite, the $X \cup Y$ is finite, Moreover, $|X \cup Y| \leq |X| + |Y|$, and if X and Y are disjoint, then $|X \cup Y| = |X| + |Y|$.
4. If S is a finite system of finite sets, then $\bigcup S$ is finite.
5. If X is finite then $\mathcal{P}(X)$ is finite.

- The addition of the Axiom of Infinity is necessary in order to obtain infinite sets.

Countable sets

The Axiom of Infinity provides us, through inductive sets, with an infinite set – the set $\mathbb{N}$ of all natural numbers.

Definition
A set S is **countable** if $|S| = |\mathbb{N}|$.
A set S is **at most countable** if $|S| \leq |\mathbb{N}|$.

- A set S is countable if it is the range of an infinite sequence $\langle a_i \mid i \in \mathbb{N} \rangle$.

e.g. $|\{2k \mid k \in \mathbb{N}\}| = |\mathbb{N}|$

Theorem
An infinite subset of a countable set is countable.

- A set is at most countable if and only if it is either finite or countable.

Countable sets

Some properties of countable sets

1. The range of an infinite sequence $\langle a_i \rangle_{i=0}^{\infty}$ is at most countable.
   - The image of a countable set under any mapping is at most countable.
2. The union of two countable sets is a countable set.
3. The union of a finite system S of countable sets is countable.
   - The union of a countable system S of countable sets is countable (requires the axiom of choice).
4. If X and Y are countable, then $X \times Y$ is countable.
5. Let $\langle A_n \mid n \in \mathbb{N}$ be a countable sytem of at most countable sets, and let $\langle a_n \mid n \in \mathbb{N} \rangle$ be a system of enumerations of A_n. Then $\bigcup_{n=0}^{\infty} A_n$ is at most countable.

Countable sets

Integers

The set $\mathbb{Z}$ of all integers is a countable set:

$$\mathbb{Z} = \{0, 1, 2, \ldots\} \cup \{-1, -2, -3, \ldots\}.$$

Rational numbers

The set $\mathbb{Q}$ of all rational numbers is a countable set.

$$f(p, q) = \frac{p}{q}, \quad q \neq 0$$

where f is one-to-on mapping from $\mathbb{Z} \times (\mathbb{Z} \times -\{0\})$ onto $\mathbb{Q}$.

Countables

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$$

Uncountable sets

From Cantor's diagonal argument we know that there are sets whose "size" is bigger than that of $\mathbb{N}$. A set S is **uncountable** if it is not at most countable.

Theorem

The set of all sets of natural number is uncountable; in fact,

$$|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|.$$

Proof.