Set Theory

Lecture 4

natural numbers; axiom of infinity; induction and recursion

Recap

First-order theory of sets

- $\mathcal{L}_\in$
- The Axioms of Existence, Extensionality, Pairing, Union, Powerset.
- The Axiom Schema of Comprehension (separation).

Sets as foundations of mathematics

- Various mathematical notions can be reduced to the notion of set.
  - ordered pairs, relations, functions, partial orderings
  - more to be defined: natural numbers (rational numbers, real numbers), operations of arithmetic, etc.
- The axioms so far introduced are sufficient to deal with elementary arithmetic and finite sets. What about infinite sets?

Infinite sets

We can talk about natural numbers (to be defined rigorously below) and their properties without having to quantify over the set of all natural numbers. But, intuitively, the notion of real number cannot be developed by means of finite sets only.

Cantor's diagonal argument (1891)

Suppose that the real numbers can be **enumerate**:

$$
\begin{aligned}
a_1 &= 0.a_{1,1}a_{1,2}a_{1,3}\dots a_{1,n}\dots \\
a_2 &= 0.a_{2,1}a_{2,2}a_{2,3}\dots a_{2,n}\dots \\
a_3 &= 0.a_{3,1}a_{3,2}a_{3,3}\dots a_{2,n}\dots \\
&\vdots \\
a_n &= 0.a_{n,1}a_{n,2}a_{n,3}\dots a_{n,n}\dots \\
&\vdots
\end{aligned}
$$

One can construct a number a^* that is not on the list above.

Natural numbers

The "natural" entry point to the realm of infinite sets is the set of **natural numbers**.

- We define/characterize the natural numbers in terms of sets.
- Then declare the set of natural numbers exists.

Dedekind Infinite

Before ZFC, Dedekind believed that he had already "proved" the existence of infinite sets.

- A set X is **Dedekind-infinite** if there is a bijective function that maps X onto some proper subset Y of X.
- Dedekind's "proof"
  - 'What are numbers and what should they be' (1888)
  - T = the set of all "objects of thinking."
  - For any $t \in T$, the thought 't is an object of thinking' (call it t^*) itself is an object of thinking, hence $t^* \in T$.
  - T_0 = the set of all thoughts of the form 't is an object of thinking', and hence T_0 is a proper subset of T.

Natural numbers

Set theoretic characterization

$0 = \varnothing$

$1 = \{0\} = \{\varnothing\}$

$2 = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$

$3 = \{0, 1, 2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

$4 = \{0, 1, 2, 3\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$

$\vdots$

$n = \{0, 1, 2, 3, \ldots, n-1\}$, ect.

In this definition, observe that $2 = \{0, 1\}$, to get 3 we adjoin a third element, namely 2, to 2 itself:

$3 = \{0, 1, 2\} = 2 \cup \{2\} = \{0, 1\} \cup \{2\}$

Similarly,

$4 = 3 \cup \{3\} = \{0, 1, 2\} \cup \{3\}$

Natural numbers

Definition

The **successor** of a set x is the set $S(x) = x \cup \{x\}$.

- Intuitively, the successor $S(n)$ of a natural number n is the "next" or the "one bigger" number $n + 1$.
- We often use the more suggestive notation '$n + 1$' for $S(x)$ when we talk about natural numbers.

N.B. The definition of successors are not restricted to sets that correspond to natural numbers. It applies to all sets.

- Question: how to isolate the set of all natural numbers in the ocean of sets?

Definition (inductive set)

A set X is said to be **inductive** if

i. $\varnothing \in X$.

ii. If $x \in X$, then $S(x) \in X$

Natural numbers

Definition

The set of all natural numbers is the set

$$\mathbb{N} = \bigcap \{X \mid X \text{ is inductive}\}$$

? Verify that $\mathbb{N}$ itself is uniquely determined and inductive.

- The existence of $\mathbb{N}$ can be justified by the axiom schema of comprehension *only if there is an inductive set at all*!

Axiom of Infinity: There exists an inductive set

in symbols: $\exists Y[\varnothing \in Y \wedge \forall x \in X(x \cup \{x\} \in X)]$

Next, we demonstrate that our set-theoretically defined natural numbers behave in the familiar ways.

Definition

The relation $<$ on $\mathbb{N}$ is defined by : $m < n$ iff $m \in n$.

Properties of natural numbers

The Induction Principle

Let $P(x)$ be a property (possibly with parameters). Assume that

(a) $P(0)$ holds.

(b) For all $n \in \mathbb{N}$, $P(n)$ implies $P(n + 1)$.

Then P holds for all natural numbers.

Proof.

Lemma

1. *$0 \leq n$, for all $n \in \mathbb{N}$.*
2. *For all $k, n \in \mathbb{N}$, $k < n + 1$ if and only if $k < n$ or $k = n$.*

Proof.

Theorem

$(\mathbb{N}, <)$ *is a (strictly) linearly ordered set.*

Properties of natural numbers

The Induction Principle (second version)

Let $P(x)$ be a property. Assume that, for all $n \in \mathbb{N}$,

if $P(k)$ holds for all $k < n$, then $P(n)$.

Then P holds for all natural numbers.

Definition (well-ordering)

A linear ordering $\prec$ of a given set A is a **well-ordering** if every nonempty subset of A has a least element. The pair $(A, \prec)$ is called a **well-ordered set**.

- The notion of well-ordering is a backbone of set theory. We will study it extensively when we define ordinal numbers.

Theorem

$(\mathbb{N}, <)$ *is a well-ordered set.*

Proof.

Recursion theorem

The next order of business is to define addition, multiplication, and other familiar operations of arithmetic in our theory.

Definition (sequences)

A **sequence** is a function whose domain is either a natural number or $\mathbb{N}$.

Finite sequences of length n: $\langle a_i \mid i < n \rangle$ or $\langle a_0, a_1, \ldots, a_{n-1} \rangle$.

Infinite sequences: $\langle a_i \mid i \in \mathbb{N} \rangle$ or $\langle a_i \rangle_{i=0}^{\infty}$.

Example

1. The sequence $s : \mathbb{N} \to \mathbb{N}$ is defined by
   $s_0 = 1$;
   $s_{n+1} = n^2$, for all $n \in \mathbb{N}$.
2. The sequence $f : \mathbb{N} \to \mathbb{N}$ is defined by
   $f_0 = 1$;
   $f_{n+1} = f_n \times (n + 1)$, for all $n \in \mathbb{N}$.

Recursion theorem

The Recursion Theorem

For any set A, and any $a \in A$, and any function $g : A \times N \to A$, there exists a unique infinite sequence $f : \mathbb{N} \to A$ such that

(a) $f_0 = a$;

(b) $f_{n+1} = g(f_n, n)$, for all $n \in \mathbb{N}$.

- The proof of this theorem is non-trivial.
- This theorem allows us to define many arithmetic operations.

Corollary

There is a unique function $+ : \mathbb{N} \to \mathbb{N}$ *such that*

(a) $+(m, 0) = m$ *for all* $m \in \mathbb{N}$*;*

(b) $+(m, n + 1) = +(m, n) + 1$*, for all* $m, n \in \mathbb{N}$*.*