Set Theory

Lecture 3

axiom of separation; relations, functions, and ordering

Recap

First-order theory of sets

The language $\mathcal{L}_\in$ of axiomatic set theory is a standard FOL (with identity) with one two-place predicate $\in$.
We have so far introduced

Axiom of Existence: $\exists x \forall y (y \notin x)$.

Axiom of Extensionality: $\forall X \forall Y [\forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y]$.

Axiom of Pairing: $\forall x \forall y \exists Y \forall u [u \in Y \leftrightarrow (u = x \vee u = y)]$.

Axiom of Union: $\forall X \exists Y \forall y [y \in Y \leftrightarrow \exists z (y \in z \wedge z \in X)]$.

Axiom of Powerset: $\forall X \exists Y \forall y [y \in Y \leftrightarrow y \subseteq X]$.

The following are verifiable in they system so far constructed

- $\exists ! \varnothing$
- For any set X of sets, $\bigcup X = \bigcup \{S \mid S \in X\}$ and $\mathcal{P}(X)$ exist.
- Let $a, b, c \in X$, show that $\{a, b, c\}$ exists.
- ? Let A, B be sets, can you justify $C = A \cap B$ exist?

The axiom schema of comprehension

The axioms of pairing, union, and powerset have an *expansive* function inasmuch as they yield the existences of sets. Yet, their applications are still quite limited.

- Let X contains all the natural numbers, isolate the subset of X that contains all the even numbers in the current theory.
- We are in need of a *restrictive* operation which allows to "scale down" sets.

Axiom Schema of Comprehension: Let $\varphi(x, u)$ be a wff. For any X and u, there exists a set $Y = \{x \in X \mid \varphi(x, u)\}$

in symbols: $\forall X \forall u \exists Y [x \in Y \leftrightarrow x \in X \wedge \varphi(x, u)]$

- Also known as the axiom of **separation**
  - Compare to the unrestricted axiom of comprehension
- It's an axiom schema instead of an axiom.
- History: Zermelo (1908) $\Rightarrow$ Fraenkel (1921/22) $\Rightarrow$ Skolem.

The axiom schema of comprehension

Axiom Schema of Comprehension:
$$\forall X \forall u \exists Y [x \in Y \leftrightarrow x \in X \wedge \varphi(x, u)]$$

Theorem

Let A and B be two sets, then there exists the set of the members that belong to both A and B (i.e., $A \cap B$ exists).

Proof.

The axiom schema of comprehension

Axiom Schema of Comprehension:

$$\forall X\ \forall u\ \exists Y[x \in Y \leftrightarrow x \in X \wedge \varphi(x, u)]$$

- A "negative" consequence: there is no set which contains all sets.
- $X - Y = \{x \in X \mid x \notin Y\}$
- $X \triangle Y = X - Y \cup Y - X$
- It implies the existence of the empty set: $\varnothing = \{x \mid x \neq x\}$
- In general, let X be an (nonempty) system of sets, then $\bigcap X$ exists
- Get used to notations $\bigcup$ and $\bigcap$ and their definitions

? We know that $\bigcup \varnothing = \varnothing$, what about $\bigcap \varnothing$?

Sets and mathematics

The axioms (schema) we have introduced so enable us to define some basic mathematical concepts and notions within set theory.

Definition (ordered pairs)

$(a, b) = \{\{a\}, \{a, b\}\}$

- ordered vs. unordered pairs.
- its coordinates can be "read off" from the definition.
  - even for the case (a, a)
- the current definition was given by Kuratowski (1921), publicized through Bourbaki's textbook on set theory.
  - Wiener (1914): $(a, b) = \{\{\{a\}, \varnothing\}, \{b\}\}$
  - Hausdorff (1914): $(a, b) = \{\{a, 1\}, \{b, 2\}\}$
  - Or we can have $(a, b) = \{\{a, b\}, \{b\}\}$
- But the choice of the definition is not entire arbitrary, the current definition is convenient to define other basic mathematical concepts – it is taken as a **primitive notion**.

Relations

Justify the definition of ordered pairs is well defined.

Theorem

$(a, b) = (a', b')$ *if and only if* $a = a'$ *and* $b = b'$.

Proof.

Relations

Mathematician often study **relations** R between mathematical object — one of the most basic concepts in mathematics. Relations can be completely defined in set theory by ordered pairs.

Definition (relations)

A set R is a binary relation is all elements of R are ordered pairs, i.e., if for any $r \in R$ there exist x and y s.t. $z = (x, y)$

- It is customary to write xRy instead of $(x, y) \in R$. We say that x is in relation r with y if xRy holds.

e.g. $R_1 = \{(a, b), (c, d)\}$, try to write it in full.

e.g. $R_2 = \{z = (m, n) \mid$ m and n are positive integers and m divides $n\}$

Let R be a relation

Domain of R $\operatorname{dom} R = \{x \mid \text{there exists some } y \text{ s.t. } xRy\}$

Range of R $\operatorname{ran} R = \{y \mid \text{there exists some } x \text{ s.t. } xRy\}$

Relations

? Justify the domain and the range of R exist. Let $A = \bigcup(\bigcup R)$, show that for any $(x, y) \in R$ $x, y \in A$.

Let R be a relation and A, B be sets. The **image** of A under R, denoted by $R[A]$, is defined as

$$R[A] = \{y \in \operatorname{ran} R \mid \text{there exists } x \in A \text{ for which } xRy\}.$$

The **inverse image** of B under R, denoted by $R^{-1}[B]$, is defined

$$R^{-1}[B] = \{x \in \operatorname{dom} R \mid \text{there exists } y \in B \text{ for which } xRy\}.$$

? Let $A = \{1\}$ and $B = \{11\}$ and R_2 as above, what are $R[A]$ and $R^{-1}[B]$?

The **inverse** of R, denoted by R^{-1}, is defined as

$$R^{-1} = \{z \mid z = (x, y) \text{ for some } x \text{ and } y \text{ such that } (y, x) \in R\}.$$

Relations

Let R and S be binary relations. The **composition** of R and S is the relation $S \circ R$ is defined by

$$S \circ R = \{(x, y) \mid \text{there exists some } z \text{ s.t. } (x, z) \in R \text{ and } (z, y) \in S\}$$

Let A and B be sets, the **Cardesian product** of A and B, denoted by $A \times B$, is defined by

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

To justify $A \times B$ exists, we need the axioms of union, powerset, and separation:

$$A \times B = \{z \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid z = (x, y) \text{ and } x \in A \text{ and } y \in B\}.$$

Functions

Definition

A binary relation F is called **function** (or mapping) if

$$xFy_1, xFy_2 \Rightarrow y_1 = y_2 \qquad \text{for any } x, y_1, y_2.$$

Let F be a function and A and B be sets.

1. F is a function **on** A if $\operatorname{dom} F = A$.
2. F is a function **into** B if $\operatorname{ran} F \subseteq B$.
3. F is a function **onto** B if $\operatorname{ran} F = B$.
4. The **restriction** of F to A is the function

$$F \restriction A = \{(x, y) \in F \mid x \in A\}.$$

N.B. We often use lower case letters $f, g, \ldots$ for functions.

5. A function f is said to be **injective** (or **one-to-one**) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$).

Orderings

Let R be a binary relation in A, where A is a set.
R is **reflexive** in A if, for all $x \in A$, xRx.
R is **symmetric** in A if, for all $x, y \in A$, xRy and yRx.
R is **asymmetric** in A if, for all $x, y \in A$, xRy implies $\neg yRx$.
R is **transitive** in A if, for all $x, y, z \in A$, xRy and yRz imply xRz.
R is **an equivalence** in A if it is reflexive, symmetric, and transitive in A.
R is **antisymmetric** in A if, for all $x, y \in A$, xRy and yRx implies $x = y$.

Definition

A binary relation R in A is said to a **(partial) ordering** if it is reflexive, antisymmetric, and transitive. It is a **strict** ordering if it is asymmetric and transitive. (A, R) is called an **ordered set**.

- The subset relation $\subseteq_A$ (w.r.t A).
- The predicate of 'is an ancestor of'

N.B. Notationally, we often use $\leq$, $<$, $\preccurlyeq$, $\prec$ for partial orderings.

Orderings

Let $\leq$ be an ordering on A and $x, y \in A$. Say that
x and y are **comparable** w.r.t $\leq$ if either $x \leq y$ or $y \leq x$ holds.
x and y are **incomparable** if neither $x \leq y$ nor $y \leq x$ holds ($x \bowtie y$).
(Both definitions can be stated in terms of strict ordering $<$).

- The sets of even and odd number under $\subseteq_{\mathbb{N}}$
- You and your cousin under the relation of 'is an ancestor of'

An ordering $\leq$ in A is said to be **total** or **linear** if any two elements of A are comparable. That is, for any $x, y \in A$, either $x \leq y$ or $y \leq x$. The pair $(A, \leq)$ is called a linearly ordered set.

- The natural numbers are linearly ordered under the "natural ordering".

Let $(A, \leq)$ be a partially ordered set and $B \subseteq A$, B is a **chain** in A is any two elements of B are comparable.

- Think about your family tree and the branch you are on.

Orderings

Let $(A, \leq)$ be a partially ordered set, and $B \subseteq A$,
a is a **maximal** element of B if $a \in B$ and $\neg(\exists x \in B) a < x$;
a is **the greatest** element of B if $a \in B$ and $(\forall x \in B) x \leq a$.

- The greatest element of B is also a maximal element.
- If A is linearly ordered, then every maximal element of B is also the greatest.

a is a **minimal** element of B if $a \in B$ and $\neg(\exists x \in B) x < a$;
a is the **least** element of B if $a \in B$ and $(\forall x \in B) a \leq x$.

- The least element of B is also a minimal element.
- If A is linearly ordered, then every minimal element of B is also the least.

a is an **upper bound** of B if $(\forall x \in B) x \leq a$;
a is a **lower bound** of B if $(\forall x \in B) a \leq x$.

- The difference between the greatest element of B and an upper bound of B is that the latter does not require $a \in B$.

Orderings

a is the **supremum** of A if a is the least upper bound, denoted

$$a = \sup A;$$

a is the **infimum** of A if a is the greatest lower bound, denoted

$$a = \inf A.$$

Example

Let $(\mathbb{R}, \leq)$ be the usual (linearly) ordered set among real numbers.

$$A_1 = \{x \mid 0 < x < 1\},$$
$$A_2 = \{x \mid 0 \leq x < 1\},$$
$$A_3 = \{x \mid 0 < x\},$$
$$A_4 = \{x \mid x < 0\}.$$

Find out the great/least elements and the supremum/infimum of A_i $(i = 1, 2, 3, 4)$.

Orderings

Let $(A, <)$, $(B, \prec)$ be two (strictly) partially ordered sets, an **isomorphism** between them is an one-to-one function h mapping from A onto B such that, for any $a, a' \in A$,

$$a < a' \iff h(a) \prec h(a').$$

An isomorphism of A onto itself is called an **automorphism** of $(A, <)$.

- The function h above is sometimes referred to as an **order preserving** mapping from A to B.
- The notion of isomorphism will be come important when we study well-ordered sets and their order types (ordinal numbers).