Set Theory

Lecture 2

preliminaries; the axiomatic foundations

Re-examination of the concept of sets

Cantor's "naive" conception of sets

- The study of the inifinite
- Antinomies

Axiomatic Foundations of Set Theory

- Zermelo-Fraenkel set theory (ZF)
  - Ernst Zermelo (1908)
  - Fraenkel and Skolem in around 1922 independently proposed the axiom of replacement. So more appropriately, ZF should be referred to as Zermelo-Fraenkel-Skolem set theory.
  - The axiom of foundation (regularity) was introduced by von Neumann (1925).
  - With the addition of the axiom of choice, we have ZFC.

Preliminaries

Before we venture into axiomatic set theory, let's review some basic set theoretic notions and operations.

Membership relation $\in$

Let X be any set, x is a member of X is denoted by $x \in X$.

Ways to denote a set

1. Enumeration $\{x_1, x_2, x_3, \ldots, x_n\}$
   - This way may be ambiguous or inefficient especially when the set contains many members.
2. Description $\{x \mid \ldots x \ldots\}$
   - Describe the defining properties that the members of a set have.

   e.g. $\{x \mid x \in \mathbb{N}$ and x is divisible by $3\}$

Preliminaries

Empty set $\varnothing$

A set is an **empty set** if it contains no member.

- Early notation '$\{\}$'.
- The '$\varnothing$' notation was introduced by the Bourbaki group (specifically André Weil) in 1939.
- The set of even prime number greater than 2.

? $\varnothing \in \varnothing$

Singletons $\{x\}$

A set is said to be a **singleton** if it contains only one member.

- $\{1\}$, $\{\{1\}\}$
- $\{\varnothing\}$, $\{\{\varnothing\}\}$

? $\varnothing \in \{\varnothing\}$

? $\varnothing \in \{\{\varnothing\}\}$

Preliminaries

Subsets $\subseteq$

Let X, Y be two sets, X is said to be a **subset** of Y if every member of X is a member of Y.

$$X \subseteq Y \quad \text{if} \quad x \in X \Rightarrow x \in Y$$

X is a **proper subset** of Y, $X \subset Y$, if $X \subseteq Y$ and $X \neq Y$.

- ▸ $X \subseteq X$
- ▸ $\varnothing \subseteq X$
- ▸ If $X \subseteq Y \subseteq Z$ then $X \subseteq Z$
- ▸ $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$
- ? $\varnothing \subseteq \varnothing$
- ? $\varnothing \subset \varnothing$
- ? $\varnothing \subseteq \{\varnothing\}$

Preliminaries

Powerset

Let X be a set, then the **powerset** of X, denoted by $\mathcal{P}(X)$, is the set of all subsets of X, in symbols, $\{x \mid x \subseteq X\}$.

- ▸ Recall Cantor's theorem: every set is strictly "smaller" than its powerset.
- ▸ $\mathcal{P}(\{a\}) = \{\varnothing, \{a\}\}$
- ▸ $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
- ? $\mathcal{P}(\{1, 2, 3\}) =$
- ? $\mathcal{P}(\varnothing) =$
- ? $\mathcal{P}(\{\varnothing\}) =$
- ? $\mathcal{P}(\{\varnothing, a\}) =$

Preliminaries

Intersection $\cap$

Let X, Y be two sets, the **intersection** of X and Y, denoted by $X \cap Y$, is the set which contains **both** the member(s) of X **and** that of Y.

$$x \in X \cap Y \quad \text{iff} \quad x \in X \text{ and } x \in Y$$

or

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

Union $\cup$

Let X, Y be two sets, the **union** of X and Y, denoted by $X \cup Y$, is the set which contains **either** the member(s) of X **or** that of Y.

$$x \in X \cup Y \quad \text{iff} \quad x \in X \text{ or } x \in Y$$

or

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

Preliminaries

Some properties of $\cap$

- ▸ $X \cap X = X$
- ▸ $X \cap Y = Y \cap X$
- ▸ $(X \cap Y) \cap Z = X \cap (Y \cap Z)$
- ? $X \cap \varnothing = \varnothing$

Some properties of $\cup$

- ▸ $X \cup X = X$
- ▸ $X \cup Y = Y \cup X$
- ▸ $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
- ? $X \cup \varnothing = X$

Distributive Laws

- ▸ $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
- ▸ $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

Preliminaries

? $\varnothing \in \varnothing \cap \varnothing$, $\varnothing \in \varnothing \cup \varnothing$,

? $\varnothing \in \varnothing \cap \{\varnothing\}$, $\varnothing \in \varnothing \cup \{\varnothing\}$,

? $\varnothing = \varnothing \cap \{\varnothing\}$, $\varnothing = \varnothing \cup \{\varnothing\}$.

Differences

The **difference** of X and Y, $X - Y$, is defined as

$$X - Y = \{x \mid x \in X \text{ but } x \notin Y\}$$

The **symmetric difference** of X and Y, $X \triangle Y$, is defined as $X \triangle Y = (X - Y) \cup (Y - X)$

? $X - Y = \varnothing$ if and only if $X \subseteq Y$

? $X = Y$ if and only if $X \triangle Y = \varnothing$

? $X - \varnothing =$ $\varnothing - X =$ $X \triangle X =$

The axiomatic approach to set theory

The language of set theory

We develop axiomatic set theory in the framework of the first-order predicate calculus (FOL) with identity. The language contains only one predicate $\in$, i.e., the **membership relation**.

atomic formulas: $x \in y$, $x = y$

well-formed formulas:

$$\neg\varphi, \quad \varphi \wedge \phi, \quad \varphi \vee \psi, \quad \varphi \rightarrow \psi, \quad \exists x\varphi, \quad \forall x\varphi.$$

We use the notational convention $\varphi(x_1, \ldots, x_n)$ that all *free* variables of φ are among $x_1, \ldots, x_n$. A wff that has no free variables is called a *sentence*.

The goal is to

- build a rigorous first-order theory that encapsulate our intuitions about sets as studied in naive set theory.
- avoid trouble!

Defining sets

Now we have a formal language where we can talk formally about sets. But where shall we start?

- The subject matter is of course about sets.
- But, in light of antinomies and unfortunate applications of the unrestricted axiom of comprehension, we need to exercise extreme care when defining sets.

Let's start minimally.

Axiom of Existence: There exists a set that has no elements.

in symbols: $\exists x \forall y (\neg y \in x)$

- In other words, we postulate that (the) empty set $\varnothing$ exists.
- There are philosophical and mathematical reasons as to why we want to postulate the existence of (the) empty set.
- Intuitively, there is only one empty set, but we cannot prove this at this stage.

Defining sets

Axiom of Extensionality: If every element of X is an element of Y and every element of Y is an element of X then $X = Y$.

in symbols: $\forall X \forall Y [\forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y]$

- Affirms the **extensional** nature of our set theory, that is, each set is completely determined by its members.
- In contrast, an *intensional* characterization of a set relies not only on their members but also on the way the set is presented.
  - e.g. The set of all non-negative reals and the set of all squares of the reals.
    - They are the same set, extensionally, but not necessarily the same, intensionally, depending on one's purpose.
    - Fregean "modes of representation".

? Prove that there is only one empty set.

Defining sets

So far, we have admitted that some sets exist (at least there is the empty set), but we still need sufficiently more sets in order to

- capture the "healthy" part of naive set theory within our formal treatment;
- develop arithmetic and analysis in set theory.
- But we need to progress gradually and avoid admitting "too many" sets (remember? The unrestricted axiom of comprehension led to an inconsistent theory.)

Axiom of Pairing: For any two elements x and y, there exists the set Y which contains just x and y.

in symbols: $\forall x \, \forall y \, \exists Y \forall u [u \in Y \leftrightarrow (u = x \vee u = y)]$.

- $A = \{1\}$ and $B = \{2\}$ then there exists $C = \{\{1\}, \{2\}\}$

? Does $D = A \cup B$ exist?

Defining sets

Axiom of Union: For any set X there exists the set whose members are *the members of the members* of X.

in symbols: $\forall X \, \exists Y \, \forall y [y \in Y \leftrightarrow \exists z \, (y \in z \wedge z \in X)]$.

- This is to say the **union** of any system X of sets exists, which is denoted by $\bigcup X$.

  N.B. Possible notional ambiguity: the following two notions are equivalent: $A \cup B \cup C$ and $\bigcup\{A, B, C\}$.
- The axiom of union justifies the ordinary union operation among sets within our formal theory.

Axiom of Powerset: For any set X there exists a set whose members are all the subsets of X.

in symbols: $\forall X \, \exists Y \, \forall y [y \in Y \leftrightarrow y \subseteq X]$

write in full: $\forall X \, \exists Y \, \forall y [y \in Y \leftrightarrow \forall z \, (z \in y \rightarrow z \in X)]$