Paradoxes of decision theory

The following famous cases are not really paradoxes so much as demonstrations that real people are systematically – or at least predictably – irrational in the sense of violating the Savage axioms. Consideration of them, and the ease with which we slip into these errors – if that is what they are – helps to reveal how much content Savage's axiomatization really has.

Allais paradox

Case 1. A card is drawn at random from a shuffled pack of cards numbered 1-100. You get to choose between two gambles:

(1) Gamble 1 pays \$5M if any card numbered 1-10 is drawn and nothing if any card numbered 11-100 is drawn.
(2) Gamble 2 pays \$1M if card 1-11 is drawn and nothing if card 12-100 is drawn.

Case 2. A card is drawn at random from a shuffled pack of cards numbered 1-100. You get to choose between two gambles:

(3) Gamble 3 pays \$5M if any card numbered 1-10 is drawn, nothing if card 11 is drawn and \$1M if card 12-100 is drawn.
(4) Gamble 4 pays \$1M for certain.

Most people have G1 $\succ$ G2 *and* G4 $\succ$ G3. This is inconsistent with the Savage axioms. The gambles can be represented as acts taking appropriate events (E1-E4 in the table) to payoffs as follows:

	E1: 1-10	**E2: 11**	**E3: 12-100**
G1	5	0	0
G2	1	1	0
G3	5	0	1
G4	1	1	1

It is easy to see that the Allais preferences are inconsistent with expected utility maximization (this is theorem 4.38 on p. 42 of Yang's notes). In particular they violate the Sure Thing Principle (STP: more formally what gets violated is (P2) on p. 22 of Yang's notes, but we can see the point more clearly by considering the informal version on p. 20).

Ellsberg paradox

Case 1. An urn contains 90 balls, of which 30 are red and the remainder are black or yellow (in an unknown proportion). You get to choose between 2 gambles:

(5) Gamble 5 pays \$100 if you draw a red ball and nothing otherwise.
(6) Gamble 6 pays \$100 if you draw a black ball and nothing otherwise.

Case 2. Same urn, different draw: this time you must choose between these two gambles

(7) Gamble 7 pays \$100 if you draw a red or yellow ball and nothing otherwise

(8) Gamble 8 pays $100 if you draw a black or yellow ball and nothing otherwise

Most people have G5 $\succ$ G6 *and* G8 $\succ$ G7. Again this violates EU-maximization, as we can see from the following table:

	Red	**Black**	**Yellow**
G5	100	0	0
G6	0	100	0
G7	100	0	100
G8	0	100	100

Again we have a failure of the STP.

Zeckhauser paradox
Consider two games of Russian Roulette.

Case 1: a six-shooter contains four bullets and you're asked how much you'll pay to remove one of them before the game begins.

Case 2: a six-shooter contains two bullets, and you're asked how much you'll pay to remove *both* of them before the game begins.

Question: if money is of no value to you if you are dead, would you pay more in Case 1 or in Case 2? Almost everybody would pay *strictly more* in Case 2 but this must be irrational. (Exercise: explain why.)