ON COUNTABLY ADDITIVE SUBJECTIVE PROBABILITIES

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Abstract. The main objective of this work is to provide an in-depth analysis of the issue of finitely versus countably additive probability in Savage's theory of personal probability. The paper is divided into three main parts. First, we comment, by providing a brief historical review, on Savage's reasons for no requiring subjective probability derived in his decision model to be countably additive. It is pointed out that Savage's argument for avoiding countable additivity is inconclusive due to an oversight of set-theoretic details. In the second part, we discuss some defects of employing merely finitely additive probability measures in Savage's system. A diagnosis is then attempted which links the insufficiency of finite additivity to the failure of continuity in a rich probability space. The analyses then lead, in the third part, to the introduction of countable additivity as a formal assumption of the theory.

Keywords. subjective probability, utility theory, countable additivity, Savage postulates, non-measurable sets, large cardinals

1. Introduction

One recurring issue in the philosophy of probability concerns the additivity condition of probability measures as to whether a probability function is finitely or countably additive. This subject is particularly controversial within the subjectivist tradition where probabilities are taken to be the agent's degrees of belief and are often interpreted within a framework for rational decision making. In a section titled "Some mathematical details," Savage (1972) discusses his reasons for favoring finitely additive probability measures derived in his theory of subjective expected utility, he says,

> It is not usual to suppose, as has been done here, that *all* sets have a numerical probability, but rather a sufficiently rich class of sets do so, the remainder being considered unmeasurable . . . the theory being developed here does assume that probability is defined for all events, that is, for all sets of states, and it does not imply countable additivity, but only finite additivity . . . it is a little better not to assume countable additivity as a postulate, but rather as a special hypothesis in certain contexts. (*ibid.* p.40)

One main mathematical reason provided by Savage for not requiring the derived probability measures in his theory of personal probability to be countably additive is that, according to him, there does not exist a countably additive extension

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of the Lebesgue measure that is defined on the set of *all* subsets of the unit interval (or the real line), whereas in the case of finitely additive measures, such an extension does exist. Since events are taken to be "all sets of states" in his system (which can be interpreted as all subsets of the reals), countable additivity is ruled out because of this claimed defect.

Savage's remarks refer to the basic problem of measure theory posed by Henri Lebesgue at the turn of the twentieth century known as the *problem of measurability*. In the next section, we remark that Savage's assessment of countably additive extension of Lebesgue measure, and thereof his reasons for rejecting countable additivity, was in fact inconclusive due to an oversight of set-theoretic details. We illustrate this point by way of a brief historical review of Lebesgue's measure problem and some of its further developments. The goal is to situate our critical assessment of Savage's mathematical arguments for favoring finite additivity in the historical development of the measure problem. In our discussion, we also address some common misunderstandings concerning the interpretational value of sophisticated mathematical models employed in some practical sciences like decision theory. This will be followed in Section 3 by an analysis of the imbalance between finite additivity and the rich mathematical structure employed in Savage's decision model. We see these discussions as providing sufficient reasons for introducing countable additivity to Savage-style decision models, especially for the ones with rich background settings. Thus, based on Villegas (1964), in Section 4 we propose a new postulate which introduces countable additivity into Savage's system, we then discuss the relationship between countable additivity and Savage's postulate 7 and their roles played in constructing utilities.

2. Some Set-theoretic Details

2.1. **The measure problem.** In his 1902 thesis Lebesgue raised the following question about the real line: Does there exist a measure m such that

- (a) m associates with each bounded subset X of $\mathbb{R}$ a real number $m(x)$;
- (b) m is not identically zero, i.e., $m(X) \neq 0$ for some X;
- (c) m is translation-invariant: for any $X \subseteq \mathbb{R}$ and any $r \in \mathbb{R}$ define $X + r := \{x + r \mid x \in X\}$, then $m(X) = m(X + r)$;
- (d) m is countably additive (or σ-additive), that is, if $\{X_n\}_{n=1}^{\infty}$ is a collection of pairwise disjoint bounded subsets of $\mathbb{R}$, then $m\big(\bigcup_n X_n\big) = \sum_n m\big(X_n\big)$?

Lebesgue developed his measure towards the solution to this problem and, unlike other attempts made around the same period (see Bingham, 2000), the measure developed by him, later known as the *Lebesgue measure*, was constructed in accordance with certain algebraic structure of sets of the real numbers.[1] As seen, the measure problem would be solved if it could be shown that the Lebesgue measure satisfies all the measurability conditions (a)-(d). Lebesgue's question was soon answered in negative by Vitali (1905), who showed that, with the Axiom of Choice (AC), there exist sets of real numbers that are not (Lebesgue) measurable. This means that, in the presence of AC, Lebesgue's measure is definable

[1]Let S be a nonempty set, a collection $\mathcal{F}$ of subsets of S is called a σ-algebra if (1) $\varnothing \in \mathcal{F}$ and $S \in \mathcal{F}$; (2) $X \in \mathcal{F}$ implies $S - X \in \mathcal{F}$; (3) $X_1, X_2, \ldots \in \mathcal{F}$ implies $\bigcup_n X_n \in \mathcal{F}$ and $\bigcap_n X_n \in \mathcal{F}$. In the case of the reals where $S = \mathbb{R}$, let $\mathscr{B}$ be the (*Borel*) σ-algebra generated by all the sets of the form $(a, b]$ where a, b are real numbers and $a < b$. Define a *Lebesgue measure* (over Borel sets) to be the real-valued function on $\mathscr{B}$ such that $\mu(\varnothing) = 0, \mu(\mathbb{R}) = +\infty$, and, for any $a, b \in \mathbb{R}$ with $a < b$, $\mu\big((a, b]\big) = b - a$.

only for a proper class of subsets of the reals, the remainder being considered unmeasurable.

Then a natural question to ask is whether or not there exists an *extension* of Lebesgue measure which not only agrees with Lebesgue measure on all measurable sets, but is also definable for non-measurable ones. Let us refer to this question as the *revised measure problem*. The problem gives rise to a more general question as to whether there exists a real-valued measure on any infinite set. To anticipate our discussion on subjective probability measures, let us reformulate the question in terms of probabilistic measures for some general infinite set. Let S be a (countably or uncountably) infinite set, a (probability) *measure* on S is a non-negative real-valued function μ on $\mathcal{P}(S)$ such that

i. μ is defined for all subsets of S;
ii. $\mu(\varnothing) = 0$, $\mu(S) = 1$;
iii. μ is countably additive (or σ-additive), that is, if $\{X_n\}_{n=1}^{\infty}$ is a collection of pairwise disjoint bounded subsets of $\mathbb{R}$, then

$$\mu\Big(\bigcup_{n=1}^{\infty} X_n\Big) = \sum_{n=1}^{\infty} \mu(X_n). \tag{2.1}$$

Here we distinguish two cases depending on the cardinality of S: If S contains uncountably many elements (e.g., $S = \mathbb{R}$), it is now known that an extension of Lebesgue measure exists if and only if there exists a measure on the continuum (or on any S with $|S| = 2^{\aleph_0}$) satisfying conditions (i)-(iii). Hence, the revised measure problem is solved if the latter can be answered. By referring to a result of Ulam (1930) on measures on infinite sets (cf. Footnote 5 below.), Savage gave a definitive answer that such an extension does *not* exist. One main aim of this section is to point out that, in fact, there is no straightforward answer to this question: the existence of a countably additive measure on $\mathcal{P}(S)$ that extends Lebesgue measure depends on the background set theory one chooses to work in. And this question is in close connection with the theory of large cardinals, we shall return to this point presently in Section 2.2 below.

If, on the other hand, S contains only countably many elements (e.g., $S = \mathbb{N}$), then it is interesting to note that μ cannot be both countably additive and uniformly distributed (or, for that matter, μ cannot be a measure that assigns 0 to all singletons). Indeed, let $\{s_1, s_2, \dots\}$ be an enumeration of all the elements in S. Suppose that μ is uniformly distributed on S, then it must be that $\mu(s_i) = 0$ for all $i \in \mathbb{N}$.[2] But, by countable additivity, $1 = \mu(S) = \mu\big(\bigcup_{i=1}^{\infty}\{s_i\}\big) = \sum_{i=1}^{\infty} \mu(s_i) = 0$, which is absurd. Hence there does not exist a σ-additive uniform distribution on a countable set. This, in part, is the reason why de Finetti opposed the employment of countable additivity who argues that a rational agent should believe that the tickets in a countably infinite lottery have equal chance to obtain, a view shared, to a large extent, by Savage. This leads to the suggestion of weakening the additivity condition (iii). Thus, an alternative option is to replace (iii) with the following condition.

iv. μ is finitely additive, that is, for any $X, Y \subseteq S$, if $X \cap Y = \varnothing$ then

$$\mu(X \cup Y) = \mu(X) + \mu(Y). \tag{2.2}$$

[2]We write $\mu(\{s_i\})$ as $\mu(s_i)$ for short.

It is clear that (iii) implies (iv) but not vice versa, hence this condition amounts to placing a weaker constraint on the additivity condition on subjective probabilities.

2.2. **Large cardinals and additivity conditions.** Admittedly, the employment of finitely additive probability measures has far-reaching merits. It can be shown that there does exist a finitely additive uniform distribution on S with at most countably many elements satisfying properties (i), (ii), and (iv).[3] In addition, in justifying his subjective interpretation of probability, de Finetti (1937b) showed that a rational player affords at least the possibility of avoiding exposure to a sure loss if and only if the set of betting quotients upon which the player accepts satisfies conditions (i), (ii), and (iv). More precisely, let S be a space of possible states, $\mathcal{F}$ be some algebra equipped on S, members of $\mathcal{F}$ are referred to as events. An event E *occurs* just in case $s \in E$ where s is the true state of the world. Let $\{E_1, \ldots, E_n\}$ be a finite partition of S where each $E_i \in \mathcal{F}$. Further, let $\mu(E_i)$ represent the decision maker's degree of belief in the occurrence of E_i, which is manifested, it is assumed, in her betting behavior that $\mu(E_i)$ is the rate at which the decision maker is willing to enter a bet which is dependent on the occurrence of E_i with a payoff of c_i for a cost of $c_i\mu(E_i)$, where c_i is decided by the opponent and is either positive or negative. The decision maker is said to be *coherent* if there is no selection of $\{c_i\}_{i=1}^n$ by the opponent such that $\sup_{s\in S}\sum_{i=1}^n c_i\big(\chi_{E_i}(s) - \mu(E_i)\big) < 0$, where χ_{E_i} is the characteristic function of E_i. In other words, the agent's subjective probability assignments are coherent if no sequence of bets can be arranged by the opponent that yields uniformly negative returns for the bettor regardless which state of the world actually obtains. Guided by this coherence principle, de Finetti showed that there exists at least one measure μ which admits that, for any selection of payoffs $\{c_i\}_{i=1}^n$, $\sup_{s\in S}\sum_{i=1}^n c_i\big(\chi_{E_i}(s) - \mu(E_i)\big) \geq 0$. In addition, it was shown by de Finetti (1930a) that μ can be extended to any collection of events containing $\mathcal{F}$, in particular, μ can be extended to $\mathcal{P}(S)$ so that condition (i) is satisfied; and that μ is a finitely additive probability measure, that is to say, μ satisfies (ii) and (iv). According to Regazzini (2013), these mathematical results developed by de Finetti in the 1920-30s played an important role in shaping his view on the issue of additivity, from which he concluded that there is *no need* to insist on countable additivity for probability measures on an algebra of events.[4]

Savage referred to the above conviction of de Finetti's (as well as a similar result given by Banach (1932)) as part of his mathematical reasons not to impose countable additivity. Before proceeding any further, let us recapitulate the main reasons that led Savage to take that "it is a little better not to assume countable additivity as a postulate":

1. There does not exist a countably additive uniform distribution over integers, whereas in the case of finite additivity such a distribution does exist.
2. According to Savage, there does not exist a countably additive extension of the Lebesgue measure to all subsets of the reals, whereas in the case of finite additivity such an extension does exist.

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[3] Appendix B contains an example of finitely additive uniform distribution over the integers.

[4] The first chapter of de Finetti (1937b, English translation as de Finetti (1937a)) contains a nontechnical summary of de Finetti (1930a, 1931, in Italian) on "the logic of the probable" where the aforementioned mathematical results were given (see Regazzini, 2013). Given that our current focus is on Savage's system, we shall not delve further into the discussion on de Finetti's reasons for rejecting countable additivity, for recent discussions, see Williamson (1999); Regazzini (2013).

For reason (1), Savage provided no further explanation about the relationship between subjective probabilities derived in his decision-theoretic model and uniform distribution over integers. He only briefly mentioned that many of us do have a strong intuitive tendency to regard such a distribution as necessary (p.43). Yet, it seems that Savage is not entirely consistent on this point: In defending his P6, Savage explicitly points out that the approach adopted by de Finetti (1937b) and Koopman (1940a,b, 1941) is implausible in that these authors postulate that the state space can be partitioned into arbitrarily many equally probable events. He maintains that (as implied by his P6) each cell of such a partition should be arbitrarily "small" but not necessarily equally probable (p.38). However, it is unclear as to why equal partition suddenly becomes permissible when it comes to natural numbers.

Further, even we grant that uniform distribution over integers be admissible in Savage-style system, it can be shown that finite additivity is still insufficient in bringing about a coherent system with rich background settings: in Section 3 we shall discuss a classical example, involving finitely additive uniform distribution over integers, which leads to the making of a money pump. More precisely, following the works of Adams (1962) and Wakker (1993), among others, we point out that the doctrine that (subjective) probability measure be merely finitely additive may stand in violations of various rationality constraints that are commonly adopted in decision-theoretic models.

Now, let us turn to reason (2). Savage (1972, p.41) cites the well-known result of Ulam (1930) testifying that any atomless σ-additive extension of Lebesgue measure to *all* subsets of the unit interval is incompatible with the continuum hypothesis (CH), from which he concludes that there is no extension that satisfies *all* of (i)-(iii). However, it is unclear as to why this constitutes a sufficient reason for rejecting countable additivity.[5]

As a matter of fact, in his article titled "a model of set theory in which every set of reals is Lebesgue measurable" Solovay (1970) showed that such a countably additive extension of Lebesgue measure to all sets of reals *does* exist if the existence of an inaccessible cardinal (I) and a weaker version of AC, i.e. the principle of dependent choice (DC), are assumed.[6] Thus, it seems that insofar as the possibility of obtaining a σ-additive extension of Lebesgue measure to all subsets of

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[5] Ulam (1930) proved that, for any uncountable set S with $|S| = \kappa$, it can be shown in ZFC that if κ is a successor (and hence a regular) cardinal (e.g., $\kappa = \aleph_1$), then there does not exist a measure on S satisfying all of (i)-(iii). It follows that if there is a σ-additive non-trivial extension of Lebesgue measure on $2^{\aleph_0}$ then CH must fail. (It is worth mentioning that, prior to Ulam, Banach and Kuratowski (1929) showed that if there is a measure on $2^{\aleph_0}$ then $2^{\aleph_0} > \aleph_1$.) Yet, even without the concern for CH, there is an aspect of Ulam's original results that was not addressed by Savage: it was shown by Ulam (1930) that, in ZFC, if there is a σ-additive non-trivial measure μ on any uncountable set S with $|S| = \kappa$ then μ is a measure on κ such that

(1) either κ is a measurable cardinal (and hence an inaccessible cardinal), on which a nontrivial σ-additive two-valued measure can be defined;
(2) or κ is a real-valued measurable cardinal (and hence a weakly inaccessible cardinal) such that $\kappa \leq 2^{\aleph_0}$, on which a nontrivial σ-additive atomless measure can be defined.

In the second case, it is plain that μ can be extended to a measure on $2^{\aleph_0}$: for any $X \subseteq 2^{\aleph_0}$, let $\mu(X) = \mu(X \cap \kappa)$. This leads to a general method of obtaining a countably additive measure on all subsets of the reals that extends Lebesgue measure (see Jech, 2003, p. 131).

[6] The relative consistency proof given by Solovay (1970) showed that if ZFC+I has a model then ZF+DC has a model in which every set of reals is Lebesgue measurable (see also Jech, 2003, p.50).

the reals is concerned, Savage's set-theoretic argument, which calls for immediate exclusion of countable additivity, is inconclusive! For the existence of such an extension really depends on the background set theory: does not exist in ZFC+CH, but does exist in ZF+DC (assuming ZFC+I is consistent).

2.3. **The underlying mathematical structure.** In view of the set-theoretic details above, a natural reaction one may have is that it seems that, in order to determine whether or not to invoke countable additivity in Savage's personal decision theory, further investigation into the "appropriate" set theory that one should work with needs to be made. However, to provide a theoretic reason for choosing one set of set-theoretic axioms over another is no easy matter, which has become an exceedingly involved issue within the proper subject of set theory.[7] At this point, one might be puzzled that the task of isolating the "correct" axioms of large cardinals, although highly interesting as a subject of its own, might be of remote philosophical significance in revealing the role of additivity played within decision-theoretic models, especially when it comes to the development of a personal decision theory of a more "mundane kind," which was what Savage initially set forth to do. Hence, in defending finite additivity, one might be tempted to appeal to some practical reasons in setting up the appropriate mathematical details contending that countable additivity should be ruled out precisely because the cost from set-theoretic complications is high, whereas in the case of finite additivity such complications can be largely minimized.

Well, this suggested roundabout approach of circumventing set theory by suspending countable additivity would be celebrated if the strategy of restricting to merely finitely additive measures could serve the full purpose of keeping Savage's system in good health, which, as we will see, is not the case. We shall return to this point in Section 3.

Meanwhile, we would like to remark on a perhaps more important point: it would be a mistake to think that the issues associated with considering complicated mathematical constraints in modeling subjective probabilities are of merely internally mathematical interest and have little conceptual importance in uncovering the philosophical understanding of the notion of probability and hence one shall avoid these constructions when possible. This prejudice against the use of sophisticated formalism is based on a misconception about the role of mathematical models. One should bear in mind that there is an indisputable distinction between a mathematically sophisticated model of probability and the commonsense understanding of probability by "men in the street." But there is no conflict between having a good grasp of an intuitive concept on the one hand and forming a fine theory about it on the other, the latter often serve as an illuminating guide in achieving the former. And, we stress, this is particularly so when it comes to the subjective interpretation of probabilities. Here we are referring to what is usually called the constructive realist view on subjective probabilities, that is, numerical subjective probabilities are real and meaningful only in that they can be *constructed* through some representation theorem. The latter requires a fully developed theory in which various properties of probabilities can be materialized. Hence, far from being "of merely internally mathematical interests", the detailed

[7]See Gaifman (2012) and Koellner (2013) for pertinent discussions.

mathematical construction is in fact the backbone of the philosophical articulation of probabilistic subjectivism, without which the latter is a mere empty promise.

2.4. **Logical omniscience and non-measurable sets.** To be sure, Savage's set-theoretic argument for not requiring countable additivity was given in ZFC+CH, where it is known that, in the case of uncountable state space, there is no non-trivial measure satisfies simultaneously conditions (i) - (iii) above; and Savage's immediate reaction was to revise the third, i.e., the additivity condition, and restrict the attention to finitely additive measures.[8] The particular set-theoretic argument Savage relied on, namely the existence of Ulam matrix (which leads to the non-existence of a measure over $\aleph_1$, see Footnote 5), uses AC in an essential way. The latter allows for the construction of non-measurable sets in ZFC. Now if one insists on maintaining the first measurability condition of defining subjective probability measure for *all* sets of states, this amounts to introducing non-measurable sets into Savage's decision-theoretic model as representing certain events. Yet, it is unclear as to what one can benefit from making such a high demand.

Note that non-measurable sets are meaningful only insofar as we have a good understanding of the contexts in which they apply. These sets are highly interesting within certain branches of mathematics largely for the reason that their introduction reveals in a deep way the complex structures of the underlying mathematical systems. This however does not mean that these peculiar set-theoretic constructions should be carried over to a system that is primarily devised to guide rational decision making.

Here, one might respond by pointing out that, as a normative theory, the current decision framework is designed only for *idealized* Bayesian decision makers, not actual ones, hence it should be within the bounds of our super agents to conceive non-measurable sets on which subjective probabilities be defined. This line of response, we stress, misses the point of theoretical idealization. Indeed, in various highly idealized prescriptive decision-theoretic models, including Savage's original system and our analysis in later sections, it is assumed that the decision makers are equipped with extraordinary abilities. What we usually expect from these super agents are their supreme logical ability and exceptional computational capacity. However it shall be emphasized that this step of idealization is not based on a blind leap of faith, it is grounded in our understanding of the basic logical apparatus and computational processes involved. We acknowledge that, as actual reasoners, our inferential performances are bounded by various physical limitations which prevent us from reaching too far. Yet a good grasp of the underlying logical machinery gives rise to the conceivable picture as to what it means for a logically omniscient agent to fulfill, at least *in principle*, the

[8] Savage is not alone in holding this view. Seidenfeld (2001) listed the non-existence of a non-trivial, σ-additive measure defined over the power set of an uncountable set as the first of his six reasons for considering theory of finitely additive probability. It is interesting to note that Seidenfeld also referred to the result of Solovay, however no further discussion on the significant of this result on additivity was given (see Seidenfeld, 2001, p.168). See Bingham (2010, §9) for a discussion and responses to each one of Seidenfeld's six reasons. Our set-theoretic argument presented here, in response to Seidenfeld's first, i.e., the measurability reason, is different from Bingham's "approximation" argument.

task of drawing all the logical consequences from a given proposition.[9] This justificatory picture, which is based on an apparent inductive reasoning, becomes increasingly blurry when we start asking our super agent to contemplate the intriguing nature of non-measurable sets in the context of rational decision making. The Banach-Tarski paradox sets just the example of how much we lack geometric and physical intuition when it comes to non-measurable sets. That means, unlike logical omniscience, we don't even have a clear idea as to what to demand from our super agent: beyond any specific set-theoretic context there is just no good intuitive basis for jumping to non-measurable sets. Hence, it seems that if there is any set-theoretic oddity to be avoided in a personal decision theory it should be non-measurable sets.[10]

On this matter, we should add that Savage himself is fully aware that the set theoretical framework under which his personal decision model is being developed exceeds what one can expect from a rational decision maker who uses subjective probabilities to encompass the best courses of actions and fend against incoherency (Savage, 1967). He also cites the Banach-Tarski paradox as an example to show the extent to which highly abstract sophisticated set theory can contradict common sense intuitions. However, it seems that Savage's readiness to include as events "all sets of states" overwhelms his willingness to avoid this set-theoretic oddity. In practice, he takes the set of *all* subsets of the state space, the power set of the continuum in the case of the reals, as the background algebra, on which subjective probabilities are defined. We however will go with a different approach. In fact, the situation can be largely simplified if we choose to work, instead of with *all* subsets of the state space S, but with a sufficiently rich collection of subsets of S (for instance, the Borel sets $\mathscr{B}$ in the case where $S = \mathbb{R}$) where, as a well established theory, countable additivity is in perfectly good health. That is to say, instead of (i), we require that

v. μ is defined on (Lebesgue) measurable sets of S.

Note that the price of forfeiting the demand of defining probability measures on the set of all subsets of S is a rather small one to pay. It amounts to disregarding all those events that are defined by Lebesgue non-measurable sets. Indeed, even Savage himself conceded that

[9]With these being said, we should however also point out that the demand to have a deductively closed system remains as a challenge to any normative theory of beliefs. In his essay titled "difficulties in the theory of personal probabilty," Savage (1967, p.308) remarked that the postulates of his theory imply that one should behave in accordance with the logical implications of all that she knows, which can be very costly. In other words, conducting logical deductions is a very resource consuming activity, the merits it brings can sometimes be offset by its high costs. Hence some might say that the assumption of logical omniscience, a promise of the discharge of unlimited deductive resources, may at its best be seen as an unfeasible idealization. Here is not the place to open a new discussion on the legitimacy of logical omniscience. The point we are trying to make is rather that, unlike non-measurable sets, being logically all powerful, however unfeasible, is not something that is not conceptually entertainable.

[10]In an unpublished work, Haim Gaifman made a similarly point against the often cited analogy between finitely additive probabilities in countable partitions and countably additive probabilities in uncountable partitions in the literature (see, e.g., Schervish and Seidenfeld, 1986) that such an analogy plays a major heuristic role in set theory but provides no useful guideline in the case of subjective probabilities for the reason that certain mathematical structures requied to make salient this analogy has no meaning in a personal decision theory.

> All that has been, or is to be, formally deduced in this book concerning preferences among sets, could be modified, mutatis mutandis, so that the class of events would not be the class of all subsets of S, but rather a Borel field, that is, a σ-algebra, on S; the set of all consequences would be measurable space, that is, a set with a particular σ-algebra singled out; and an act would be a measurable function from the measurable space of events to the measurable space of consequences. (Savage, 1972, p.42)

It shall be emphasized that this modification of definition of events from, say, the set of all subsets of $(0,1]$ to all the Borel set $\mathscr{B}$ of $(0,1]$ is not carried out at the expense of disregarding a large collection events that are otherwise representable. As noted by Billingsley (2012, p.23), "[i]n fact, $\mathscr{B}$ contains all the subsets of $(0,1]$ actually encountered in ordinary analysis and probability. It is large enough for all 'practical' purposes." Here, by "practical purposes" we take as meaning that all events and measurable functions considered in economic theories in particular are definable using only measurable sets, and, consequently, there is no need to appeal to non-measurable sets. We shall proceed with our discussions of Savage's subjective utility representation theory in this spirit, where we require, unless otherwise specified, that all derived subjective probability measures be defined on the set of measurable sets of the state space and that they are countably additive, that is, they satisfy conditions (ii), (iii), and (v).

To sum up, Savage chose to base his theory of subjective expected utility on finitely additive probability measures. This is because he took, for various mathematical considerations, that countable additivity is *insufficient* for the constraints he set for his decision model, whereas finite additivity, it is said, is sufficient for those purposes. These led him to think that it is *not necessary* to invoke the countable additivity condition, "I know," says Savage, "of no argument leading to the requirement of countable additivity" (*ibid.* p.43). The discussion we have made so far suggests that Savage's insufficiency claim against countable additivity is inadequate due to an oversight of set-theoretic details. The main aim of the next section is then to provide a much needed necessity argument for the requirement of countable additivity in Savage's system: in Section 3, we point out that the restriction to merely finitely additive measures may render Savage's theory incoherent. These considerations will lead in Section 4 to the introduction of countable additivity as a formal component of the theory. We then discuss various attempts to extend utility function for simple acts to acts in general under Savage's P7 and countable additivity.

3. Finite and Countable Additivity in Savage's system

Let us refer to the thesis that (subjective) probabilities be finitely but *not* countably additive as *strict finitism on additivity*. Note that this is not to be confused either with strict finitism (or ultrafinitism) or finitism in philosophy of mathematics. By "strict finitism on additivity" we mean the anti-pluralist position towards additivity that probability measures can only be finitely additive in all mathematical systems in which they apply. It is plain that probability measures are finitely additive in a finitary mathematical system, the dispute lies in whether subjective probabilities shall be merely finitely additive in a given infinitary system. Clearly, Savage does not question the use of infinitary mathematics. In fact many aspects of his theory (examples, proofs, etc.) use infinity in an essential way. Yet efforts were made by him to justify for finite additivity in his decision model leaving countable additivity "as a special hypothesis in certain contexts." In this section, we argue that in many ways Savage's own decision-theoretic framework is a context in which countable additivity should apply. We proceed by illustrating that strict finitism on additivity may lead to some discomforts in Savage's system.

3.1. **Converting to countable additivity and failure of dominance.** As noted above, the probability measure derived in Savage's subjective expected utility theory features, among other things, the following properties:

Finitely Additivity: The probability measure μ on the state space S is finitely additive.

All-inclusiveness: Events are defined as all subsets of S, not a collection of events measurable with respect to some σ-field of subsets of S smaller than $\mathcal{P}(S)$.

Atomlessness: The state space S is (uncountably) infinite and the probability μ is atomless.[11]

We point out an interesting property in systems that feature these three properties: it can be shown that for each disjoint sequence $\{A_i\}$ for which countable additivity fails there exists another sequence $\{C_i\}$ such that, for each i, the probability of A_i agrees with that of C_i while countable additivity holds for $\{C_i\}$. (The proof is given in Appendix C).

Proposition 3.1. Let $\{A_i\}_{i=1}^{\infty}$ be any sequence of disjoint subsets of S, if μ is finitely additive, universal, and atomless then $\sum_{i=1}^{\infty}\mu(A_i) \neq \mu(\bigcup_{i=1}^{\infty} A_i)$ implies that there exists a partition $\{C_i\}_{i=0}^{\infty}$ of S satisfying

(1) $\mu(\bigcup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty}\mu(A_i)$;
(2) $\mu(C_i) = \mu(A_i)$ for all $i = 1, 2, \ldots$;
(3) $\sum_{i=1}^{\infty}\mu(C_i) = \mu(\bigcup_{i=1}^{\infty} C_i)$.

In light of Proposition 3.1, it seems that, *as far as probability calculus is concerned*, there is little point in restricting to merely finitely additive measures in a system that features all three properties above. For, given any finitely (but not countably) additive measure satisfying all three properties, the proof above shows that, for every sequence of events that fails countable additivity, there exists another sequence of events that has the same local probabilistic profile as the original sequence (i.e. condition (2) above) such that countable additivity holds under the same measure!

It may be objected that even though each C_i agrees probabilistically with A_i for all i's, this does not mean that $\{A_i\}$ can be replaced by $\{C_i\}$: they are after all different sequences of events and hence may lead to different configurations/partitions of the state space and different constructions of Savage acts. On this view, Proposition 3.1 can at most be seen as a peculiar feature of the system while maintaining that $\{A_i\}$ and $\{C_i\}$ shall remain as distinctive strings of events with different probabilistic properties. However, it shall be pointed out that the

[11]More precisely, a set A is said to be an *atom* of a given measure μ if, for every $X \subseteq A$, either $\mu(X) = 0$ or $\mu(A - X) = 0$. The measure μ is *atomless* if there are no atoms. The atomlessness property is implied by the following stronger condition derivable in Savage's system: for any $A \subseteq S$ and any $0 \leq \rho \leq 1$, there exists $B \subseteq A$ such that $\mu(B) = \rho\mu(A)$, (cf. Savage, 1972, p.34, Theorem 2). In the following, it is this stronger condition we will be referring to as the atomlessness condition.

admission of sequences of events with the same local but different global probabilistic properties opens the door to various counter-intuitive examples, which can be constructed using precisely the imbalance between their different probabilistic characterizations.

To name one such example, note that, in the proof of Proposition 3.1, if let A_0 denote $S - \bigcup_{i=1}^{\infty} A_i$, it is easily seen that the proposition implies that for any partition $\{A_i\}_{i=0}^{\infty}$ for which countable additivity fails there exists a partition $\{C_i\}_{i=0}^{\infty}$ such that the following holds: (1) $\mu(A_0) < \mu(C_0)$; (2) $\mu(A_i) = \mu(C_i)$, for all $i = 1, 2, \ldots$; (3) $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) > \mu\big(\bigcup_{i=1}^{\infty} C_i\big)$. Using partitions like these, Wakker (1993) constructed two Savage acts that stand in clear violation of the principle of (strict) *Stochastic dominance*.[12] The latter, however, is an intuitive rationality principle which is intimately related to Savage's sure-thing principle. It would be scandalous if this principle is violated in Savage's system.[13]

3.2. **Money pump.** Prior to Wakker, Adams (1962) showed that there are scenarios in which the failure of countably additivity leads to a money pump. More precisely, Adams' example presents a betting situation where a (Bayes) rational gambler is justified in accepting, with a small fee, each bet of a sequences of bets, but the acceptance of all the bets leads to sure loss. For illustrative purpose, we reproduce this example here.[14]

Example 3.2. Let $S = \mathbb{N}$, $X = [-1, 1]$, and let the identity function $u(x) = x$ be the utility function on X. Let λ be the finitely but not countably additive measure on positive integers given in Example B.1 (in Appendix B) and let η be the countably additive probability measure on S given by

$$\eta(n) = \frac{1}{2^n} \quad \text{for all } n \in S.$$

Define subjective probability μ to be such that

$$\mu(n) = \frac{\lambda(n) + \eta(n)}{2} \quad \text{for all } n \in S. \tag{3.1}$$

The following is a list of simple properties of μ.

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[12](Strict) Stochastic dominance mandates that act f is strictly preferred to act g if the probability of f yielding any consequence that is "at as valuable as" x is no less than that of g for all consequences $x \in X$ and with some strict inequality at some x:

$$\left.\begin{array}{l} \mu[f \succsim x] \geq \mu[g \succsim x] \text{ for all } x \in X \\ \mu[f \succsim x] > \mu[g \succsim x] \text{ for some } x \in X \end{array}\right\} \implies f \succ g.$$

Assuming that consequences are linearly ordered with endpoints, Wakker (1993) showed that there are f and g constructed, respectively, from $\{A_i\}$ and $\{C_i\}$ such that f strictly dominates g even though they have the same expected utility.

[13]The issue addressed here is closely related to what is called the *conglomerability* property for probability measures first discovered by de Finetti (1930b). Without delving too far into this literature, we should briefly mention that Hill and Lane (1985) confirmed de Finetti's conjecture that a probability measure is conglomerable if and only if it is countably additive. Schervish and Seidenfeld (1986) showed that the failure of countable additivity (for the cases with countable partitions) may lead to the failure of dominance. That is, if μ is not countably additive, then there exist two acts f and g such that f is conditionally preferred to g for each cell of the partition $\mathcal{H} = \{H_1, H_2, \ldots\}$, yet f is not unconditional preferred to g. This stands in clear violation of Savage's "loose" version of the sure-thing principle (cf. Savage, 1972, p.22). So it seems that not requiring countable additivity can be a somewhat self-undermining strategy for Savage in setting up his decision model, for it may lead to violations of one of his most fundamental postulates for rational decision making.

[14]The following example is altered from Stinchcombe (1997).

TABLE 3.1. Gamble g_n

	$B_n = \{n\}$	B_n^C
g_n	$1/2^{n+1} - r$	$1/2^{n+1}$

(1) μ is a finitely but not countably additive probability measure.
(2) For any $n < \infty$,

$$\mu(n) = \frac{0 + \eta(n)}{2} = \frac{1}{2^{n+1}}.$$

(3) $\mu(S) = 1$, whereas

$$\sum_{i=1}^{\infty} \mu(i) = \frac{\sum_{i=1}^{\infty} \lambda(i) + \sum_{i=1}^{\infty} \eta(i)}{2} = \frac{0+1}{2} = \frac{1}{2}.$$

(4) For any finite $E \subseteq S$,

$$\mu(E) = \frac{0 + \eta(E)}{2} = \sum_{i \in E} \frac{1}{2^{i+1}}.$$

Now, for each n, consider the gamble g_n with payoff described as in the matrix in Table 3.1. That is, g_n pays constantly $1/\left(2^{n+1}\right)$ no matter which state obtains, but will cost the gambler $r \in (\frac{1}{2}, 1)$ in the event of $B_n = \{n\}$ (see Table 3.1). Since $r < 1$, it is easy to calculate that, for each n, g_n has a positive expected utility:

$$U[g_n] = \frac{1}{2^{n+1}} \cdot \left(\frac{1}{2^{n+1}} - r\right) + \left(1 - \frac{1}{2^{n+1}}\right) \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} \cdot (1 - r).$$

Hence, a (Bayes) rational gambler should be willing to pay a small fee ($< U[g_n]$) to accept each gamble. However, the acceptance of all gambles leads to sure loss no matter which number eventually transpires. To see this, note that for any given m gamble g_n pays

$$g_n(m) = \frac{1}{2^{n+1}} - r \cdot \chi_{B_n}(m)$$

But, the joint of all g_n's yields

$$\sum_{i \in S} g_i(m) = \sum_{i \in S} \left[\frac{1}{2^{i+1}} - r \cdot \chi_{B_i}(m)\right] = \frac{1}{2} - r < 0.$$

That is to say, for each possible outcome $m \in S$, the expected value of getting m from accepting all the gambles g_n's is negative. $\lhd$

3.3. **Infinite bets.** Adams' money pump is surprising, because it results in a series of *incoherent* choices made by the gambler which is precisely what systems like Savage's is devised to prevent.

In dealing with this difficulty, advocates of finite additivity often argue that given that a subjective theory is a systemization of coherent decision making by rational decision makers it is unclear as to what it means for a decision maker to fulfill the task of coherently choosing *infinitely* many times. Hence, on this view, the challenge from Example 3.2, which requires the gambler to accept infinitely many gambles, is really a non-starter, for it envisages a situation that is not conceptually admissible within a finitary framework with respect to coherent choosing which is best captured by finite additivity.[15]

We however have a different view on accepting infinitely many gambles. As noted before, there is a great deal of idealizations built in the current decision-theoretic framework where we grant our decision maker with unlimited computing capacity. Again, this step of idealization is grounded in our understanding of the basic logical and computational apparatuses involved. The process of idealization is then the process of disregarding the limitations of our performances as actual reasoners. Admittedly, being deductively all-capable does not necessarily imply that our super agent has the ability of handling infinitely many gambles; besides, as we have discussed in the case of non-measurable sets, not all forms of idealization are meaningful for the purpose of developing a normative theory for personal decision making. However, it seems that the same inductive justification for the assumption of logical omniscience can be employed to provide a conceptual basis for extending the decision problem to include infinite gambles. Note that the procedure involved in deciding whether or not any given gamble g_n is acceptable is well understood (i.e., calculating its expected utility). The process of idealization is then the process of disregarding the sheer quantity of gambles a (Bayes) rational decision maker is obliged to accept, an ability our super agent is expected to possess. Unlike non-measurable sets, this process is not something that we, *qua* actual reasoners, cannot conceive. It would be a *double standard* to insist that the decision maker should be infinitely capable when it comes to computational and inferential performances, but to appeal to physical or psychological realism when it comes to accepting bets.

Perhaps, the finitist's rebuttal is due to a confusion between the physical realization of gambles and the conceptual idealization of them. It shall be clear that a normative theory of rational decision-making may draw initial intuitions from the former in setting up the basic structure of the model, but will eventually land the conceptual theorization in the latter. The success of such a theory however is not tied to the physical realizability of the decision situation it depicts. Thus, it is by virtue of this step of idealization based on certain inductive reasoning like in the case of logical omniscience we say that it is meaningful to entertain the possibility of accepting an infinite sequence of gambles.

Further, if we restrict our attention to Savage's system, it is not difficult to see that there is nothing, in both the basic formal setups and the statements of the postulates, prevents us from considering infinite sequences of gambles. As known, there are two *types* of axioms postulated in Savage's subjectivist theory. The first lays out various fundamental rationality principles governing rational decision making; the second consists of a series of structural axioms which pave the way for the eventual mathematical representation. Clearly, Adams' example stands in no violation with any of the rationality axioms. As for the structural axioms, it shall be noted that it is no more difficult in conceptualizing the acceptance of infinitely many gambles than it is to imagine that the state space can be arbitrarily divided so that, between any two acts with strict preference of one over another, infinitely many acts can be inserted on the preference scale (from which infinite sequences of gambles can be constructed). The latter is a key feature of

[15]We thank professor Isaac Levi for pointing out this line of objection, which echoes de Finetti's position that a rational agent is obliged to accept no more than finitely many fair bets at any time.

Savage's system mandated by the structural axioms. Hence, it seems that the finitists' rejection of accepting infinite gambles cannot be made without making fundamental revisions of Savage's infinitary decision model.

4. Countable Additivity and Utility Extension

4.1. **Quantitative and qualitative continuities.** Note that, in Savage's representation theorem, it is crucial that the underlying algebra is closed under infinite unions and intersections. Indeed, Savage himself remarks that

> It may seem peculiar to insist on σ-algebras as opposed to finitely additive algebras even in a context where finitely additive measures are the central objects, but countable union do seems to be essential to some theorems of §3 . . . (Savage, 1972, p.43)

This "peculiar" feature implies that, in a system with finitely additive measures, it is not always the case that the convergence of an infinite sequence of events at the limit point can be characterized in the corresponding probabilistic terms due to failure of continuity. To be more precise, let $(S, \mathcal{F}, \mu)$ be a measure space, μ is said to be *continuous from below*, if, for any sequence of events $\{A_n\}_{n=1}^{\infty}$ and event A in $\mathcal{F}$, $A_n \uparrow A$ implies that $\mu(A_n) \uparrow \mu(A)$; it is *continuous from above* if $A_n \downarrow A$ implies $\mu(A_n) \downarrow \mu(A)$, and it is *continuous* if it is continuous from both above and below.[16] It can be easily shown that continuity fails in general, if μ is merely finitely additive. As an illustration, it is interesting to note that the strictly finitely additive measure λ (Example B.1) used in constructing the probability measure in Adams' example is neither continuous from above nor from below. In fact, it can be proved that aforementioned properties of continuity hold if and only if μ is countably additive.

Intuitively speaking, in order to establish continuity it is necessary that the set functions in question (in our case derived probability measures) be sensitive to the set operations of infinite union and intersection at limit points. This balance is undermined if we allow, on the one hand, infinite unions and intersections but do not require measures to be countably additive on the other. This gives rise to the mismatch between accepting infinitely many gambles and the corresponding probability calculation, which ultimately is what fueled Adams' money pump.

One way to introduce countable additivity (and hence continuity) to Savage's system is to strengthen the underlying qualitative probability. Following Villegas (1964), let $\succeq$ be a qualitative probability, defined on a σ-algebra $\mathcal{F}$ of the state space S, $\succeq$ is said to be *monotonely continuous* if, given any sequence of events $A_n \uparrow A$ ($A_n \downarrow A$) and any event B,

$$A_n \preceq B\,(A_n \succeq B) \text{ for all } n \quad \Longrightarrow \quad A \preceq B\,(A \succeq B). \tag{4.1}$$

Moreover, Villegas showed that if a qualitative probability $\succeq$ is atomless and monotonely continuous then the numerical probability μ that agrees with $\succeq$ is unique and countably additive.[17] Since the qualitative probability measures in

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[16] As notational conventions, $A_n \uparrow A$ means that $A_1 \subseteq A_2 \subseteq \cdots$ and $\bigcup_i A_i = A$, and $\mu(A_n) \uparrow \mu(A)$ means that $\mu(A_1) \leq \mu(A_2) \leq \cdots$ and $\mu(A_n) \to \mu(A)$ as $n \to \infty$. Similar for the other case.

[17] Villegas (1964) showed that monotone continuity is a necessary and sufficient condition for the agreeing numerical measure to be countably additive. It was further shown that any qualitative probability defined on a finitely additive algebra can be extended to a qualitative probability σ-algebra statisfying monotone continuity, fineness, tightness. Thanks largely to Savage's P6, the qualitative

Savage's system are non-atomic, it is sufficient to introduce the the property of monotone continuity in order to introduce countable addivitity. We thus propose the following postulate, P8, to be added to Savage's P1-7 (cf. SVG 1-7 in Appendix A), which is a reformulation of (4.1) in terms of preferences among Savage acts.

P8: For any $a, b \in X$ and for any event B and any sequence of events $\{A_n\}$,
(i) if $A_n \uparrow A$ and $\mathfrak{c}_a \oplus_{A_n} \mathfrak{c}_b \precsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$ for all n then $\mathfrak{c}_a \oplus_A \mathfrak{c}_b \precsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$;
(ii) if $A_n \downarrow A$ and $\mathfrak{c}_a \oplus_{A_n} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$ for all n then $\mathfrak{c}_a \oplus_A \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$.

Here, we have taken the step of introducing countable additivity as a formal assumption to Savage's decision model through an added postulate. Countable additivity, however, cannot replace the role played by P7. In the next section, we investigate the relations between countable additivity and P7 in extending utilities from simple acts to general acts.

4.2. **Countable additivity and P7.** Savage (1972, p.78) gave an example which satisfies the first six of his seven postulates but not the last one. This is intended to show that the seventh postulate (SVG 7) is independent from other postulates in Savage's original system. Upon showing the independence of P7, Savage remarked that "[f]inite, as opposed to countable, additivity seems to be essential to this example," and he conjectured that "perhaps, if the theory were worked out in a countably additive spirit from the start, little or no counterparts of P7 would be necessary." This section is aimed at providing a deeper analysis of Savage remark on the relation between countable additivity and utility extension under various versions of P7. Let us start with the footnote Savage added to the remark above: "Fishburn (1970, Excercise 21, p.213) has suggested an appropriate wakening of P7." It turned out that this is inaccurate. To wit, the following is Fishburn's suggestion (expressed using our notation).

P7b: For any event $E \in \mathcal{F}$ and $a \in X$, if $\mathfrak{c}_a \succsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$ then $\mathfrak{c}_a \succsim_E g$; and if $\mathfrak{c}_a \precsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$ then $\mathfrak{c}_a \precsim_E g$.

P7b is weaker than SVG 7 in that it compares act g with a constant act instead of another general act f. Note that Fishburn's P7b is derived from the following condition A4b occurred in his discussion on preferences axioms and bounded utilities (*ibid.* §10.4).

A4b: Let X be a set of prizes/consequences and $\Delta(X)$ be the set of all probability measure defined on X, then for any $P \in \Delta(X)$ and any $A \subseteq X$ if $P(A) = 1$ and, for all $x \in A$, $\delta_x \succsim (\precsim)\delta_y$ for some $y \in X$ then $P \succsim (\precsim)\delta_y$, where δ_x denotes the probability that degenerates at x.

A4b, together with other preference axioms discussed in the same section, are used to illustrate, among other things, the differences between measures that are countably additive and those are not. It was proved by Fishburn that the expected utility hypothesis holds under A4b, that is,

$$P \succ Q \iff E(u, P) > E(u, Q), \quad \text{for all } P, Q \in \Delta(X). \tag{4.2}$$

if $\Delta(X)$ contains *only* countably additive measure. Fishburn then showed, by way of a counterexample, that the hypothesis fails if the set of probability measure contains also merely finitely additive ones. Because of its direct relevancy to our

probabilities derived in the system are atomless, fine, and tight, then countable additivity obtains if the monotone continuity is in place.

discussion on the additivity condition, let us reproduce this example (Fishburn, 1970, Theorem 10.2) here.

Example 4.1. Let $X = \mathbb{N}^+$ with $u(x) = x/(1+x)$ for all $x \in X$. Let $\Delta(X)$ be the set of all probability measures on the set of all subsets of X and defined u on $\Delta(X)$ by

$$u(P) = E(u, P) + \inf\Big\{P\big[u(x) \geq 1 - \epsilon\big] : 0 < \epsilon \leq 1\Big\}. \tag{4.3}$$

Define $\succ$ on $\Delta(X)$ by $P \succ Q$ iff $u(P) > u(Q)$. It is easy to show that A4b holds under this definition. However if one takes P to be the measure in Example B.1, i.e., a finitely but not countably additivity probability measure, then we have $u(\lambda) = 1 + 1 = 2$. Hence $u(\lambda) \neq E(u, \lambda) = 1$. This shows the expected utility hypothesis fails under this example. ◁

However, as pointed by Seidenfeld and Schervish (1983, Appendix), Fishburn's proof of (4.2) using A4b was given under the assumption that $\Delta(X)$ is closed under countable convex combination (condition S4 in Fishburn, 1970, p.137), which in fact is not derivable in Savage's system. They show through the following example (Example 2.3 in Seidenfeld and Schervish, 1983, p.404) that the expected hypothesis fails under the weakened P7b (together with SVG 1-SVG 6) and this is so even when the underlying probability be countably additive.

Example 4.2. Let S be $[0, 1)$ and X be the set of rational numbers in $[0, 1)$. Let μ be uniform probability on measurable subsets of S and let all measurable function f from S to X satisfying $V[f] = \lim_{i\to\infty} \mu\big[f(s) \geq 1 - 2^{-i}\big]$ be acts. For any act f, let $U[f] = \int_S u(f)d\lambda$ where $u(x) = x$ is a utility function on X and define

$$\begin{aligned} W[f] &= U[f] + V[f] \\ &= \int_S u(f(s))d\mu(s) + \lim_{i\to\infty} \mu\Big[f(s) \geq 1 - \frac{1}{2^i}\Big]. \end{aligned} \tag{4.4}$$

Further, define $f \succ g$ if $W[f] > W[g]$. It is easy to see that SVG 1-SVG 6 are satisfied. To see that W satisfies P7b, note that if for any event E and any $a \in X$, $\mathfrak{c}_a \succsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$, then by (4.4), we have $1 > u(a) \geq u(g(s))$ for any $s \in E$. Note that $1 > u(g(s))$ implies $V[g\chi_E] = 0$ where χ is the indicator function. Thus, $W[\mathfrak{c}_a\chi_E] = \int_E u(a)d\mu \geq \int_E u(g(s))d\mu(s) = W[g\chi_E]$. The case $\mathfrak{c}_a \precsim_E \mathfrak{c}_{g(s)}$ can be similarly shown. ◁

In other words, contrary to what Savage had thought, P7b is in fact insufficient in bringing about a full utility representation theorem even in the presence of countable additivity. This shows, *a fortiori,* that countable additivity alone is insufficient in carrying the utility function derived from SVG 1-SVG 6 from simple acts to general acts.[18] Seidenfeld and Schervish (1983, Example 2.2) also showed that this remains the case even the set of probabilities measure is taken to be closed under countable convex combination.

As seen, Savage's last postulate which plays the role of extending the utilities from simple acts to acts in general cannot be easily weakened even in the presence of countable additivity. Yet, on the other hand, it is clear that countable additivity is a stronger condition than finite additivity originally adopted in Savage's theory.

[18] I thank professor Teddy Seidenfeld for helping me see this point.

So, for future works, one might be interested in finding an appropriate weakening of P7 in a Savage-style system augmented with countable additivity.

5. Concluding Remarks

The debate about finite versus countable additivity often operates on two fronts. First, there is the concern for mathematical consequences as to whether or not the kind of additivity in use accords well with demanded mathematical details. Second, there is the concern for philosophical interpretations as to whether or not the kind of additivity in use is conceptually justifiable (using our intuitions or rationality principles like the Dutch book argument etc.).

Regarding the first concern, Savage provided a set-theoretic argument against countable additivity arguing that countable additivity, it is said, is not in good conformity with demanded set-theoretic details. As we have seen in Section 2, this line of reasoning is unwarranted: his argument is misguided due to an overlook of some crucial technical details concerning large cardinals assumptions and non-measurable sets. Countable additivity *can* be coherently incorporated without serious set-theoretic complications.

As for the second concern with conceptual justification, we have seen above that there are cases in which finite additivity is insufficient in bringing about a coherent theory in Savage's framework due to the failure of continuity on the part of derived probability measures. This is largely due to the imbalance between finite additivity and the rich background settings employed in his system, which naturally led us to enlist countable additivity as an additional postulate.

Indeed, as far as Savage's system is concerned, there is no sufficient reason as to why the decision model has to be restricted to just finitely additive measures. In fact, as suggested in the opening quote of this chapter, there is no need to be dogmatic about this additivity condition, countable additivity can be employed as needed. Yet, on the other hand, given how widespread countably additive measures are used in modern probability theory, it is advantageous to presuppose countable additivity and only to weaken this condition when it is called for.

Appendix A. Savage's Postulates

Let S be the set of states of the world and X the set of consequences. A Savage act f is a function mapping from S to X. An act is said to be a *constant act* with respect to consequence $a \in X$, denote by $\mathfrak{c}_a$, if $\mathfrak{c}_a(s)$ for all $s \in S$. Define the *combination* of acts f and g with respect to an event E (a set of states), written $f \oplus_E g$, to be such that:

$$(f \oplus_E g)(s) = \begin{cases} f(s) & \text{if } s \in E \\ g(s) & \text{if } s \in E^C, \end{cases} \tag{A.1}$$

where $E^C = S - E$ is the compliment of E. Below is a list of Savage's postulates, which are equivalent to Savage's **P1-7**.

SVG 1. $\succsim$ is a weak order (complete preorder) among Savage acts.

SVG 2. For any $f, g \in \mathcal{A}$ and for any $E \subseteq S$, $f \succsim_E g$ or $g \succsim_E f$.

SVG 3. For any $a, b \in X$ and for any non-null event $E \subseteq S$, $\mathfrak{c}_a \succsim_E \mathfrak{c}_b$ if and only if $a \succsim b$.

SVG 4. For any $a, b, c, d \in X$ satisfying $a \succsim b$ and $c \succsim d$ and for any events $E, F \subseteq S$, $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_F \mathfrak{c}_b$ if and only if $\mathfrak{c}_c \oplus_E \mathfrak{c}_d \succsim \mathfrak{c}_c \oplus_F \mathfrak{c}_d$.

SVG 5. For some constant acts $\mathfrak{c}_a, \mathfrak{c}_b \in \mathcal{A}$, $\mathfrak{c}_b \succ \mathfrak{c}_a$.

SVG 6. For any $f, g \in \mathcal{A}$ and for any $a \in X$, if $f \succ g$ then there is a finite partition $\{P_i\}_{i=1}^n$ such that, for all i, $\mathfrak{c}_a \oplus_{P_i} f \succ g$ and $f \succ \mathfrak{c}_a \oplus_{P_i} g$.

SVG 7. For any event $E \in \mathcal{F}$, if $f \succsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$ then $f \succsim_E g$.

Appendix B. Uniform Distribution over the Natural Numbers

Example B.1. Let $\{\lambda_n\}$ be a sequence of functions defined on $\mathbb{N}$ such that

$$\lambda_n(i) = \begin{cases} 1/n & \text{if } 1 \le i \le n, \\ 0 & \text{if } i > n. \end{cases} \tag{B.1}$$

It can be shown that there exists a function λ defined for all subsets of $\mathbb{N}$ that extends d.[19] Furthermore, λ satisfies the following properties:

(1) λ is defined for all subsets of $\mathbb{N}$.
(2) $\lambda(\varnothing) = 0$ and $\lambda(\mathbb{N}) = 1$.
(3) λ is finitely additive.
(4) λ is *not* countably additive.
(5) For any $i < \infty$, $\lambda(\{i\}) = 0$.
(6) For any $A \subseteq \mathbb{N}$, if A is finite then $\lambda(A) = 0$; if A is cofinite (i.e. if $\mathbb{N} - A$ if finite) then $\lambda(A) = 1$.
(7) $\lambda(\{2n \mid n \in \mathbb{N}\}) = 1/2$, i.e., the set of even numbers has measure $1/2$.
(8) In general, the set of numbers that are divisible by $m < \infty$ has measure $1/m$, that is, $\lambda(\{1m, 2m, 3m, \ldots\}) = 1/m$. As a result of this property, we have that the assignment of μ can be arbitrarily small: for any $\lambda > 0$, there exists some n such that the set of numbers that are divisible by n has measure $1/n < \epsilon$. $\lhd$

Appendix C. Proof

Proof of Proposition 3.1. Suppose that $\{A_i\}_{i=1}^\infty$ is a sequence of pair-wise disjoint subsets of S such that $\sum_{i=1}^\infty \mu(A_i) \neq \mu(\bigcup_{i=1}^\infty A_i)$. The latter implies that exactly one of $<$ or $>$ holds. By finite additivity, $\sum_{i=1}^n \mu(A_i) = \mu\big(\bigcup_{i=1}^n A_i\big) \le \mu\big(\bigcup_{i=1}^\infty A_i\big)$. So it can only be that

$$\sum_{i=1}^\infty \mu(A_i) < \mu\Big(\bigcup_{i=1}^\infty A_i\Big). \tag{C.1}$$

By the atomlessness property above, there exists some $B \subseteq S$ such that

$$\mu(B) = \sum_{i=1}^\infty \mu(A_i). \tag{C.2}$$

Similarly, there exists some $B_1 \subseteq B$ for which $\mu(B_1) = \mu(A_1)$. By finite additivity, we have $\mu(B - B_1) = \mu(\bigcup_{i=2}^\infty A_i)$. Using the same argument, there exist some

[19] See Rao and Rao (1983, Theorem 3.2.10) for a version of the existence result (Kadane and O'Hagan (1995, Theorem 1) show that the monotonicity condition given by Rao and Rao (1983) in their extension theorem is also necessary, see also Schirokauer and Kadane (2007).) Hrbacek and Jech (1999, Ch. 11) contains a set-theoretic construction.

$B_2 \subseteq B - B_1$ for which $\mu(B_2) = \mu(A_2)$. Repeat this process until we form a sequence $\{B_i\}_{i=1}^{\infty}$ of subsets of B such that $B_{i+1} \subseteq B - \bigcup_{j=1}^{i} B_j$ and $\mu(B_i) = \mu(A_i)$ for all $i = 1, 2, \ldots$. This together with (C.2) yield that

$$\mu(B) \geq \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) \geq \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} \mu(A_i) = \mu(B). \tag{C.3}$$

From which we conclude, via finite additivity, that $\mu(B - \bigcup_{i=1}^{\infty} B_i) = 0$. The proof is completed if we let $\{C_i\}_{i=0}^{\infty}$ be a partition of S such that $C_0 = (S - B) \cup (B - \bigcup_{i=1}^{\infty} B_i)$ and $C_i = B_i$ for all $i = 1, 2, \ldots$. □

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