

ELEMENTS OF BAYESIAN DECISION THEORY

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CHAPTER I

Subjective Expected Utility

Let us now try to find a method of measuring beliefs as bases of possible actions The old-established way of measuring a person's belief is to propose a bet, and see what are the lowest odds which he will accept.

— Ramsey (1926)

1. Introduction

1.1. Degrees of belief and betting method. The opening quote is from Frank Ramsey's celebrated essay *Truth and Probability* (1926) where Ramsey proposed a theory of personal probability and utility. The theory contained basic ideas which were later developed or rediscovered in, most notably, the works of de Finetti, von Neumann and Morgenstern, Savage, Anscombe and Aumann, among others. Ramsey's subjectivism introduced a novel idea of measuring at the same time decision maker's subjective utilities and probabilities, where the agent's personal probability was portrayed as "the logic of partial beliefs" which was given an operational definition through coherent betting behavior. Let us highlight some basic ideas of Ramsey's theory.¹

Let X be a set of prizes and, to simplify matters, let x^* and x_* represent respectively the best and the worst prizes considered by the decision maker.² In order to get a precise measure of her subjective valuations of the prizes in X , the decision maker is presented with the following *betting* situation which involves an ethically neutral proposition p :

- i. x^* if p ; x_* if $\neg p$
- ii. x^* if $\neg p$; x_* if p

where (i) and (ii) can be seen as two lottery tickets with the redeeming policy that if, say, ticket (i) is chosen and p is indeed true then the agent will be rewarded with x^* , x_* if p is false. According to Ramsey, a proposition p is said to be *ethically neutral* "if two possible worlds differing only in regard to the truth of p are always of equal value" (cf. Ramsey,

¹Ramsey's paper was first read at the "Moral Sciences Club" at University of Cambridge in 1926 and published posthumously in *The Foundations of Mathematics and other Logical Essays* (1931), edited by Richard Braithwaite. It also appears in a collection of Ramsey's writings edited by Mellor, D. H., *Philosophical Papers*, Cambridge University Press, 1990.

²Note that Ramsey's original theory does not postulate the existence of these two distinguished prizes (in fact, any two distinctive prizes would do). We introduce x^* and x_* purely for illustrative purposes, which will be used in later expositions.

1926, p.73). In other words, the truth (or the falsity) of p itself has no added value in evaluating the value of a bet. As we shall see, this assumption is a forerunner of the state independent axiom adopted in various decision models which will be discussed later.

Now, under the above betting setup, if the decision maker is indifferent between (i) and (ii), then it is said that the agent has $1/2$ *degree of belief* in p being true. Ramsey then postulates in the form of an axiom that there exists an ethically neutral proposition p believed to degree $1/2$. This distinguished proposition p (with $1/2$ degree of belief) can be further used to evaluate the values of other prizes in X . Consider the following bets

- iii. x
- iv. x^* if p ; x_* if $\neg p$.

If the agent is indifferent between the two bets, then the value of x said to be equal to half of the total value of x^* and x_* . To represent numerically, let the utility of x^* be 1 and x_* be 0, in symbols $u(x^*) = 1$ and $u(x_*) = 0$. According to Ramsey, the fact that the decision maker's is indifferent between the two tickets implies that her evaluation of the utility of x is the midpoint of the utility scale from 0 to 1:

$$u(x^*) \text{ --- } \frac{1}{2} \text{ --- } u(x) \text{ --- } u(x_*)$$

It is further assumed that the above procedure can be repeated indefinitely, that is, for instance, there exists some prize x' whose utility halves the way from x_* to x with $u(x') = 1/4$, and so on. Hence, under this assumption, the utility scale between x^* and x_* can be calibrated to arbitrary precision. Then, for any $y \in X$, y can be assigned with a numerical utility representation $u(y)$ on the utility scale.

$$u(x^*) \text{ --- } \dots \text{ --- } \frac{1}{2} \text{ --- } u(x) \text{ --- } \dots \text{ --- } \frac{1}{4} \text{ --- } u(x') \text{ --- } \dots \text{ --- } u(x_*)$$

With subjective utilities for all prizes in hand, Ramsey proceeds to define what it means to say that the agent believes in the truth of an arbitrary proposition q to certain *degree* using the following betting mechanism. For any q , if there exist prizes $x, y, z \in X$ with $u(y) \geq u(x) \geq u(z)$ such that the agent is indifferent between the following bets

- v. x
- vi. y if q ; z if $\neg q$

then her *partial belief* in q , denoted by $\mu(q)$, is defined as

$$\mu(q) = \frac{u(x) - u(z)}{u(y) - u(z)}, \quad u(y) - u(z) > 0.$$

Using a "Dutch-book argument" Ramsey shows that if the agent's partial belief assignments are *coherent*, in the sense that no book can be made against her, then μ obeys the

laws of probability calculus (cf. [Ramsey, 1926](#), p.79).³ We will not go further into the Dutch-book argument here which is a topic on its own, for further discussion see, for instance, [Earman \(1992, Ch.2\)](#), [Hajek \(2008\)](#). Our focus is rather to see how probabilities and utilities are derived in various formal systems.

1.2. Expected utility theory. Ramsey's essay marked the beginning of a series of extensive studies in utility theory. In this and the next chapter we explore three main representation theorems. Here is a quick preview. (The readers may ignore the technical details upon first reading.)

vNM: Let X be a *finite* set of prizes/consequences, and \mathcal{L}_X be the set of probability measures on X . Each $p \in \mathcal{L}_X$ is referred to as a *lottery* on X , the intended interpretation is that, for any prize $x \in X$, $p(x)$ is the probability of getting x . Let \succsim be a preference relation on \mathcal{L}_X , the **von Neumann-Morgenstern** (vNM) expected utility theory states that if \succsim satisfies certain postulated axioms then it can be presented by a utility function (EUF) $U : \mathcal{L}_X \mapsto \mathbb{R}$ such that

$$p \succsim q \iff U(p) \geq U(q),$$

where U can be expected utilities, that is, there exists a subjective utility function $u : X \mapsto \mathbb{R}$ for which

$$p \succsim q \iff \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x),$$

and u is unique up to a positive linear transformation. We further extend this result to the case where X may contain *infinitely* many consequences and each $p \in \mathcal{L}_X$ is simple (i.e., has finite support).

A-A: Let S be a *finite* set of states of the world. Define a *horse-race lottery* to be a function h mapping from S to \mathcal{L}_X . Denote the space of horse race lotteries by \mathcal{H} . Then given any horse lottery h and state $s \in S$, $h(s)$ is a (vNM) lottery defined on X , we also write h_s for $h(s)$. Hence, for any prize $x \in X$, $h_s(x)$ is the probability that x is obtained in state s given the horse lottery h . An **Anscombe-Aumann** (A-A) representation of a preference relation \succsim on \mathcal{H} is that there exists a utility function $u : X \mapsto \mathbb{R}$ and a (subjective) probability measure μ on an algebra of S such that, for any $h, h' \in \mathcal{H}$,

$$h \succsim h' \iff \sum_{s \in S} \mu(s) \sum_{x \in X} h_s(x)u(x) \geq \sum_{s \in S} \mu(s) \sum_{x \in X} h'_s(x)u(x)$$

provided that \succsim satisfies a set of postulated axioms.

³For more detailed discussions/expositions on Ramsey's account, see [Fishburn \(1981, §5.1\)](#), [Bradley \(2001\)](#).

SVG: Further, let S be an (uncountably) *infinite* set of states and \mathcal{F} be some algebra equipped on S , X be a set of consequences, and let \mathcal{A} be the set of functions mapping from \mathcal{F} to X , each $f \in \mathcal{A}$ is referred to as an *act*. Then a **Savage** representation of the preference ordering \succsim on \mathcal{A} is that, under postulated axioms on \succsim , there exists a (subjective) probability measure μ on (S, \mathcal{F}) and a real-valued utility function u on X such that, for any $f, g \in \mathcal{A}$,

$$f \succsim g \iff \int_S u[f(s)] d\mu \geq \int_S u[g(s)] d\mu.$$

REMARK. 1. The decision-theoretic models listed here are *not* presented in chronicle order: the Anscombe-Aumann model appeared after the first edition of Savage’s *Foundations of Statistics*. The materials presented here are organized based on the methodological approach they each adopts with increasing computational complexity.

2. In the three decision models above, the respective preference relations are defined on different sets of alternatives (see Table 1.1 for a comparison). To simplify notations, we adopt a systematic ambiguity and use, unless otherwise specified, the same notation “ \succsim ” for all preference relations and let the context determine on which set of alternatives a preference relation is defined.

TABLE 1.1. Models of Expected Utility.

	\succsim on	defined	subjective utility	subjective probability	objective probability
Ramsey	\mathcal{A}^a		✓	✓	– ^b
vNM	\mathcal{L}_X		✓	–	✓
A-A	\mathcal{H}		✓	✓	✓
SVG	\mathcal{A}		✓	✓	–

^a Ramsey uses “propositions” instead of states and events and his preferences are defined for consequences, acts, and conditional acts.

^b Strictly speaking there is no objective probability explicitly employed in Ramsey’s model, yet it is easily seen that his notion of ethically neutral propositions with 1/2 degrees of belief, which is based on an apparent symmetry consideration, play a similar role as some chance mechanism.

1.3. Kinds of probability. In the discussions below, we pay close attention to different *kinds* of probability involved, by which we are referring to the (rough) distinction between *objective* and *subjective* probabilities. These probabilities may appear either as measures (subjective probability) of decision maker’s personal probabilistic judgments over the occurrences of some events or in the form of some presupposed chance mechanism (objective probability).

“Probability has often been visualized as a subjective concept more or less in the nature of an estimation. Since we propose to use it in constructing an individual, numerical estimation of utility, the above view of probability would not serve our purpose. The simplest procedure is, therefore, to insist upon the alternative, perfectly well founded interpretation of probability as frequency in the long run.” (von Neumann and Morgenstern, 1944, p.19)

The quotation is from von Neumann and Morgenstern’s well-known book “The Theory of Games and Economic Behavior,” where they made a distinction between two different kinds of probabilities. As we shall see, the use of objective probabilities is crucial to the vNM model and the A-A model, where the decision maker’s personal utilities and subjective probability (in the case of the A-A model) are essentially defined in terms of objective chances. Savage, on the other hand, adopted a purely subjective interpretation of probability upholding that subjective utility can be integrated with respect to subjective probability. The tradeoff is that Savage’s theory is considerably more complicated than other models.

The plan for this chapter is as follows. In Section 2 and Section 3, we reconstruct the decision models developed in von Neumann and Morgenstern (1944) and Anscombe and Aumann (1963). The expositions owe much to Fishburn (1970, 1981, 1986, 1994); Hammond (1998a,b); Kreps (1988); Mehta (1998); Ok (2007, 2011); Rubinstein (2007), to name just a few. Full mathematical details of the theories discussed in this chapter can be found in the works just cited, our primary goal is to trace the methodological developments that are related to Savage’s theory of subjective expected utility, which will be the main theme of the next chapter.

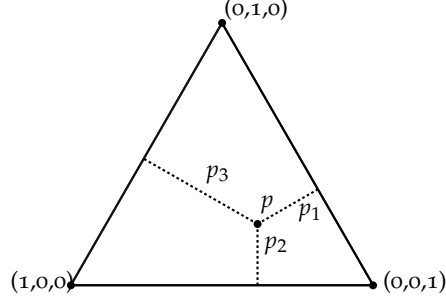
2. von Neumann-Morgenstern Utility Functions

2.1. Lotteries. As mentioned above, a *lottery* on a finite set X is a probability function p on X , we sometimes refer to p as a von Neumann-Morgenstern (vNM) lottery. The intended interpretation is that X is a set of prizes and $p(x)$ is the *chance* that $x \in X$ obtains. Let \mathcal{L}_X be the set of all probability functions on X .⁴ In simple cases, define the *degenerate lottery* with respect to any given $x \in X$ to be the probability function $\delta_x \in \mathcal{L}_X$ such that, for any $y \in X$,

$$\delta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}. \quad (2.1)$$

That is, δ_x assigns probability 1 to x , 0 otherwise. Hence each prize $x \in X$ can be identified with a lottery δ_x that degenerates at x . Then, it is easily seen that, for any

⁴ \mathcal{L}_X is often written as $\Delta(X)$, namely the space of probability functions defined on X .

FIGURE 2.1. 2-Simplex with *unit* altitude.

lottery $p \in \mathcal{L}_X$, p can be written as a combination of δ_x 's,

$$p = \sum_{x \in X} p(x) \delta_x. \quad (2.2)$$

EXAMPLE 2.1. Suppose that $X = \{x_1, x_2, x_3\}$. Write the degenerate lottery δ_{x_1} in the form of a triple $(1, 0, 0)$ which says that δ_{x_1} assigns probability 1 to prize x_1 and 0 to both x_2 and x_3 . Then \mathcal{L}_X can be represented geometrically by a 2-simplex in Figure 2.1 where each vertex of the equilateral triangle corresponds to a degenerate lottery.⁵ Note that since in an equilateral triangle the sum of the perpendiculars from any internal point p to three sides equals its altitude (say, 1), write any point p in the triangle in the form of (p_1, p_2, p_3) where each coordinate p_i is the length of the perpendicular from p to the edge that is on the opposite side of vertex i , then $p_1 + p_2 + p_3 = 1$. Thus, Figure 2.1 is a representation of the lotteries in \mathcal{L}_X where each point $p = (p_1, p_2, p_3)$ corresponds to a lottery in \mathcal{L}_X with p_i being the probability that p assigns to x_i . \triangleleft

DEFINITION 2.2. Let $p, q \in \mathcal{L}_X$, a *compound lottery* of p and q with scalar $\lambda \in [0, 1]$ is a function r such that $r(x) = \lambda p(x) + (1 - \lambda)q(x)$ for all $x \in X$. Denote the compound lottery r by the following notation,

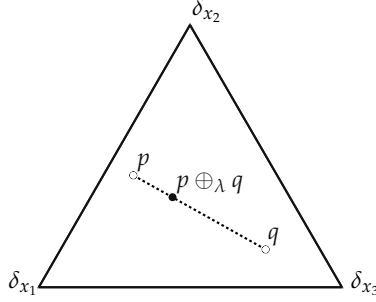
$$p \oplus_\lambda q := \lambda p(x) + (1 - \lambda)q(x). \quad (2.3)$$

Intuitively, given any $p, q \in \mathcal{L}_X$, a compound lottery $p \oplus_\lambda q$ can be considered as a (second-order) lottery ticket which has the payment policy that, with known chance λ , lottery p will transpire and, with probability $(1 - \lambda)$, lottery q obtains.

$$\begin{array}{c|cc} & \lambda & 1 - \lambda \\ \hline p \oplus_\lambda q & p & q \end{array} \quad (2.4)$$

It is easy to see that $p \oplus_\lambda q \in \mathcal{L}_X$, that is, every (second-order) compound lottery is in effect equivalent to a (first-order) lottery in \mathcal{L}_X . To characterize this concept geometrically

⁵Strictly speaking, a standard n -simplex is a unit $n + 1$ -dimensional polygon in \mathbb{R}^{n+1} , Figure 2.1 is a special case where the 2-simplex is represented as a space of its own in \mathbb{R}^2 .

FIGURE 2.2. Compound lottery $p \oplus_{\lambda} q$ in 2-simplex.

using the simplicial representation of Example 2.1, we have that the point that represents the compound lottery $p \oplus_{\lambda} q$ ($0 \leq \lambda \leq 1$) in Figure 2.2 falls on the line segment that joins p and q .

2.1.1. *Preference over lotteries.* Presumably, the decision maker has preferences over the prizes. It is assumed that these preferences are reflected in her preferences over the lotteries with each lottery specifying the chances of getting the prizes.⁶ For instance, in Example 2.1, suppose that the agent definitely prefers prize x_1 over other two prizes then it must be that she prefers δ_{x_1} over δ_{x_2} and δ_{x_3} , because the latter two lotteries assign probability 0 to the obtaining of x_1 . Formally, let \succsim be a preorder on \mathcal{L}_X (see Appendix A.1), which represents the decision maker's preferences over all the lotteries. The following are the *von Neumann-Morgenstern postulates* on \succsim :⁷

vNM 1. \succsim is a complete preference relation.

vNM 2. For all $p, q, r \in \mathcal{L}_X$ and any $\lambda \in (0, 1]$,

$$p \succ q \iff p \oplus_{\lambda} r \succ q \oplus_{\lambda} r.$$

vNM 3. For any $p, q, r \in \mathcal{L}_X$, there exist $a, b \in (0, 1)$, such that

$$p \succ r \succ q \implies p \oplus_a q \succ r \succ p \oplus_b q.$$

vNM 1 is often referred to as the *completeness* axiom which asserts that all lotteries are pair-wisely comparable. This axiom is often defended along the lines that the decision maker, if pressed, will eventually make a decision between a given pair of options regardless what her deliberation process might be. Note that given the agent's ordering among lotteries one can induce an ordering \succsim^* over the prizes through degenerate

⁶See von Neumann and Morgenstern (1964, §3.3.1) for their discussions on the relation between probabilistic reasonings and utility considerations.

⁷See Hammond (1998a, §3) for a discussion on different versions of the independence and the continuity axioms adopted in the literature. The current system (vNM 1-3) is provably equivalent to the theory presented there due to Jensen (1967) (see conditions (O), (I), (C), and Lemma 4.5(a), see also Fishburn (1977, 1981, 1982)).

lotteries as follows: for all $x, y \in X$,

$$x \succsim^* y \quad =_{\text{Df}} \quad \delta_x \succsim \delta_y. \quad (2.5)$$

That is, prize x is said to be weakly preferred to prize y , if, under the initial ordering \succsim , the degenerate lottery δ_x is at least as good as the lottery that degenerates at y . It is easily seen that if \succsim is totally (or partially) ordered so is \succsim^* . This preference relation among prizes induced through degenerate lotteries can be seen as a precursor of Savage's similar notion of preferences over consequences which is induced via the notion of constant acts defined in Definition 5.3. vNM 2 is commonly known as the *independence* axiom. To explain in terms of compound lotteries, the axiom says that decision maker's (strict) preference between two lotteries remains the same when each is combined with the same lottery (with respect to the same scalar). To illustrate, observe that the compound lotteries in (2.6) are so arranged that they agree with one another on $(1 - \lambda)$, then vNM 2 mandates that the preference between the two combined lotteries is solely determined on the part where they are different, i.e., on λ . This postulate is closely related to (or, perhaps, motivates) Savage's well known *sure-thing* principle which will be examined in §5.2.

$$\begin{array}{c|cc} & \lambda & 1 - \lambda \\ \hline p \oplus_\lambda r & p & r \\ q \oplus_\lambda r & q & r \end{array} \quad (2.6)$$

vNM 3 is sometimes called the *Archimedean* or *continuity* axiom. Intuitively, it says that no lottery p (q) is so good (bad) that, for any $r \succ q$ ($p \succ r$), the compound lottery of p and q is always better (worse) than r . Variants of vNM axioms are widely adopted in utility theory as they provide some basic characterization of the underlying preferential structure which mimics the behavior of the standard ordering \geq on the real line. The latter paves the way for the eventual real-valued numerical utility representation of \succsim . The following properties can be derived from the axioms.

LEMMA 2.3. For any $p, q, r \in \mathcal{L}_X$ and $\lambda \in (0, 1]$,

- (1) $p \sim q$ if and only if $p \sim p \oplus_\lambda q$;
- (2) $p \succsim q$ if and only if $p \succsim p \oplus_\lambda q \succsim q$;
- (3) for any $0 \leq \beta < \alpha \leq 1$, $p \succ q$ if and only if $p \oplus_\alpha q \succ p \oplus_\beta q$;
- (4) if $p \succsim r \succsim q$ and $p \succ q$, then there exists a unique $\alpha \in [0, 1]$ such that $r \sim p \oplus_\alpha q$.

PROOF. (1) Suppose, to the contrary, that $p \succ p \oplus_\lambda q$. Write p as $p \oplus_\lambda p$, then we have $p \oplus_\lambda p \succ p \oplus_\lambda q$. The latter implies, via vNM 2, $p \succ q$, a contradiction. Hence $p \oplus_\lambda q \succsim p$ by vNM 1. Similarly, it can be shown $p \succsim p \oplus_\lambda q$. Thus $p \sim p \oplus_\lambda q$.

(2) Suppose, to the contrary, that $q \succ p \oplus_\lambda q$, that is, $q \oplus_\lambda q \succ p \oplus_\lambda q$. It follows, by vNM 2, that $q \succ p$, a contradiction. It can be similarly shown that it is not the case that $p \oplus_\lambda q \succ p$. Thus, by vNM 1, $p \succsim p \oplus_\lambda q \succsim q$.

(3) If $\beta = 0$, then, by vNM 2, $p \succ q$ implies $p \oplus_\alpha q \succ q \oplus_\alpha q = q = p \oplus_\beta q$. If $0 < \beta < \alpha \leq 1$, then $1 - \beta/\alpha \in (0, 1)$, by vNM 2, $p \oplus_\alpha q \succ q$ implies that

$$\begin{aligned} p \oplus_\alpha q &= (p \oplus_\alpha q) \oplus_{1-\frac{\beta}{\alpha}} (p \oplus_\alpha q) \\ &\succ q \oplus_{1-\frac{\beta}{\alpha}} (p \oplus_\alpha q) \\ &= \left(1 - \frac{\beta}{\alpha}\right)q + \frac{\beta}{\alpha}[\alpha p + (1 - \alpha)q] = \beta p + (1 - \beta)q = p \oplus_\beta q. \end{aligned}$$

(4) The claim is trivially true if $r \sim p$ or $r \sim q$, in which cases $\alpha = 1$ or 0 , respectively. We prove the case where $p \succ r \succ q$. Consider the sets

$$\begin{aligned} A &:= \{x \in [0, 1] \mid p \oplus_x q \succsim r\}; \\ B &:= \{x \in [0, 1] \mid r \succsim p \oplus_x q\}. \end{aligned}$$

Let $\alpha_* = \inf A$ and $\alpha^* = \sup B$. Note that, for any $a > \alpha_*$, there must exist some $a' \in A$ such that $a > a' \geq \alpha_*$ (for, otherwise, a is a lower bound of A that is greater than α_* , which contradicts the assumption $\alpha_* = \inf A$), hence, by claim (3) above, $p \oplus_a q \succ p \oplus_{a'} q \succsim r$. It follows, via vNM 1, that

$$a > \alpha_* \implies a \notin B. \quad (2.7)$$

The contrapositive of (2.7) says that, for any a , $a \in B$ implies that $\alpha_* \geq a$, in other words, α_* is an upper bound of B . and hence $\alpha_* \geq \alpha^*$. Similarly, one can show that, for any a ,

$$\alpha^* > a \implies a \notin A \quad (2.8)$$

which leads to $\alpha^* \geq \alpha_*$. Now define $\alpha = \alpha_* = \alpha^*$. The proof is completed if we can show that $\alpha \in A \cap B$. Suppose, to the contrary, that $\alpha \notin B$, then, by vNM 1, $p \oplus_\alpha q \succ r$. It follows, by vNM 3 and the assumption $r \succ q$, that there exists some $a \in (0, 1)$ such that $(p \oplus_\alpha q) \oplus_a q \succ r$, that is, $p \oplus_{a \cdot \alpha} q \succ r$. This implies that $a \cdot \alpha \in A$. However, from $\alpha^* = \alpha > a \cdot \alpha$ we get, via (2.8), that $a \cdot \alpha \notin A$, a contradiction. Hence we have $\alpha \in B$. Similarly, one can show $\alpha \in A$. Uniqueness can be easily derived from (2.7) and (2.8). \square

REMARK 2.4. Note that vNM 3 can also be derived from Lemma 2.3(4) under vNM 1 and vNM 2. To see this, let p, q, r be such that $p \succ r \succ q$, we show that there exist $a, b \in (0, 1)$, such that $p \oplus_a q \succ r \succ p \oplus_b q$. By Lemma 2.3(4) there exists a unique $c \in (0, 1)$ for which $r \sim p \oplus_c q$. Then let a be any number in $(c, 1)$ and b be any number in $(0, c)$, then, by Lemma 2.3(3) (which is derivable from under vNM 1 and vNM 2), we

are done. Thus vNM 3 is provably equivalent to Lemma 2.3(4) given vNM 1 and vNM 2. For this reason, we can use vNM 3 and Lemma 2.3(4) interchangeably as the continuity axiom of vNM theory.

2.2. Cardinal utility. Given the assumption that X is *finite*, it follows, by vNM 1, that the set of degenerate lotteries $\{\delta_x \mid x \in X\}$ has a \succsim -maximal and a \succsim -minimal element, that is, there exist a most desired prize x^* and a least desired prize x_* in X such that

$$\delta_{x^*} \succsim \delta_x \succsim \delta_{x_*}, \quad \text{for all } x \in X. \quad (2.9)$$

The following lemma shows that δ_{x^*} and δ_{x_*} are in fact extreme points for all lotteries in \mathcal{L}_X under \succsim .

LEMMA 2.5. There exist $x^*, x_* \in X$ such that $\delta_{x^*} \succsim p \succsim \delta_{x_*}$ for all $p \in \mathcal{L}_X$.

PROOF. Let δ_{x^*} and δ_{x_*} be defined as in (2.9). Consider the non-trivial case where $\delta_{x^*} \succ \delta_{x_*}$. Note that, for any $p \in \mathcal{L}_X$, p can be rewritten as $\sum_{x \in X} p(x)\delta_x$ via (2.2). Then, by Lemma 2.3(4), for each δ_x , let $\lambda_x \in [0, 1]$ be such that $\delta_x \sim \delta_{x^*} \oplus_{\lambda_x} \delta_{x_*}$, hence, by vNM 2,

$$\begin{aligned} p &= \sum_{x \in X} p(x)\delta_x \sim \sum_{x \in X} p(x)(\delta_{x^*} \oplus_{\lambda_x} \delta_{x_*}) \\ &= \sum_{x \in X} p(x)\lambda_x\delta_{x^*} + \left[1 - \sum_{x \in X} p(x)\lambda_x\right]\delta_{x_*}. \end{aligned} \quad (2.10)$$

Since $\delta_{x^*} \succ \delta_{x_*}$ and $0 \leq \sum_{x \in X} p(x)\lambda_x \leq 1$, then, by Lemma 2.3(2),

$$\delta_{x^*} \succsim \sum_{x \in X} p(x)\lambda_x\delta_{x^*} + \left[1 - \sum_{x \in X} p(x)\lambda_x\right]\delta_{x_*} \sim p.$$

Similarly, it can be shown that $p \succsim \delta_{x_*}$. □

Let us now proceed with the main theorem of this section.

THEOREM 2.6 (von Neumann-Morgenstern). Let X be a nonempty finite set, and \succsim be a preference relation on \mathcal{L}_X . Then \succsim satisfies vNM 1-3 if and only if there exists a function $u \in \mathbb{R}^X$ such that

$$p \succsim q \text{ iff } \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x), \quad (2.11)$$

where u is unique up to a positive linear transformation, that is, for any function $v \in \mathbb{R}^X$, v satisfies (2.11) if and only if, for some $a > 0$ and b ,

$$u(x) = av(x) + b. \quad (2.12)$$

PROOF. We only prove the non-trivial “only if” direction of the theorem in following steps:

- (1) By Lemma 2.5, there exist $x^*, x_* \in X$ such that $\delta_{x^*} \succsim p \succsim \delta_{x_*}$ for all $p \in \mathcal{L}_X$. If $\delta_{x^*} \sim \delta_{x_*}$ then $p \sim q$ for all $p, q \in \mathcal{L}_X$. In this case, let u be any constant function. Otherwise, $\delta_{x^*} \succ \delta_{x_*}$, define function $U : \mathcal{L}_X \mapsto [0, 1]$ as follows,

$$U(p) := \inf \{ \alpha \in [0, 1] \mid \delta_{x^*} \oplus_\alpha \delta_{x_*} \succsim p \}.$$

By Lemma 2.3(3),

$$p \succsim q \text{ if and only if } U(p) \geq U(q) \text{ for all } p, q \in \mathcal{L}_X; \quad (2.13)$$

and by Lemma 2.3(4),

$$p \sim \delta_{x^*} \oplus_\lambda \delta_{x_*} \text{ if and only if } \lambda = U(p). \quad (2.14)$$

- (2) We show that U is an affine function on \mathcal{L}_X , that is, for any sequences $\lambda_1, \dots, \lambda_n \in [0, 1]$ with $\sum_i \lambda_i = 1$ and for any $p_1, \dots, p_n \in \mathcal{L}_X$, we have

$$U(\lambda_1 p_1 + \dots + \lambda_n p_n) = \lambda_1 U(p_1) + \dots + \lambda_n U(p_n). \quad (2.15)$$

It suffices to show, for any $p, q \in \mathcal{L}_X$ and $\lambda \in [0, 1]$, that

$$U(\lambda p + (1 - \lambda)q) = \lambda U(p) + (1 - \lambda)U(q). \quad (2.16)$$

Note that, by (2.14), $p \sim \delta_{x^*} \oplus_{U(p)} \delta_{x_*}$ and $q \sim \delta_{x^*} \oplus_{U(q)} \delta_{x_*}$. Let $r = \lambda p + (1 - \lambda)q$. Then, by vNM 2 (twice),

$$\begin{aligned} \delta_{x^*} \oplus_{U(r)} \delta_{x_*} &\sim r = \lambda p + (1 - \lambda)q \\ &\sim \lambda (\delta_{x^*} \oplus_{U(p)} \delta_{x_*}) + (1 - \lambda) (\delta_{x^*} \oplus_{U(q)} \delta_{x_*}) \\ &= \delta_{x^*} \oplus_{\lambda U(p) + (1 - \lambda)U(q)} \delta_{x_*}. \end{aligned}$$

It follows that $U(r) = \lambda U(p) + (1 - \lambda)U(q)$ via (2.14).

- (3) Now, for any $p, q \in \mathcal{L}_X$, by (2.13) and (2.2),

$$p \succsim q \text{ if and only if } U\left(\sum_{x \in X} p(x) \delta_x\right) \geq U\left(\sum_{y \in X} q(y) \delta_y\right)$$

Then, by (2.15),

$$p \succsim q \text{ if and only if } \sum_{x \in X} p(x) U(\delta_x) \geq \sum_{y \in X} q(y) U(\delta_y).$$

Define $u : X \mapsto \mathbb{R}$ to be such that $u(x) = U(\delta_x)$, we have that

$$p \succsim q \text{ if and only if } \sum_{x \in X} p(x) u(x) \geq \sum_{y \in X} q(y) u(y).$$

- (4) Finally, we show that u is unique up to a positive linear transformation, we prove only the nontrivial “only if” direction of the proof. As before, let $x^*, x_* \in X$ be such

that $\delta_{x^*} \succsim \delta_x \succsim \delta_{x_*}$ for all $x \in X$. Further, let a, b be such that

$$u(x^*) = av(x^*) + b, \quad u(x_*) = av(x_*) + b,$$

where $a > 0$ (the existence of such a, b is guaranteed by the hypothesis $\delta_{x^*} \succ \delta_{x_*}$ in part (1)). By (2.14), for any $x \in X$, there exists a number λ for which $\delta_x \sim \delta_{x^*} \oplus_\lambda \delta_{x_*}$, then we have that

$$\begin{aligned} u(x) &= \lambda u(x^*) + (1 - \lambda)u(x_*) \\ &= \lambda [av(x^*) + b] + (1 - \lambda) [av(x_*) + b] \\ &= a [\lambda v(x^*) + (1 - \lambda)v(x_*)] + b = av(x) + b. \end{aligned}$$

This completes the proof of the theorem. \square

We refer to the derived function u above as an instance of *von Neumann-Morgenstern utility function* (vNMUF). A pair of vNMUFs u and v are said to be *cardinally equivalent* if (2.12) is satisfied. The following corollary is a generalization of Theorem 2.6, which will become handy later. The proof uses the same techniques as in the proof Theorem 2.6, and hence omitted.

COROLLARY 2.7. Let X, \mathcal{L}_X be as above, and let C be any convex subset of \mathcal{L}_X . Suppose that \succsim is a preference relation on C such that there is an \succsim -maximum and an \succsim -minimum in C . Then \succsim satisfies vNM 1-3 if and only if there exists a (utility function) $u \in \mathbb{R}^X$ such that

$$p \succsim q \text{ iff } \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x). \quad (2.17)$$

where u is unique up to a positive linear transformation.

2.3. Expected utility for simple lotteries. We now extend the von Neumann and Morgenstern Expected Utility Theorem 2.6 to a class of lotteries defined for some X that contains potentially *infinitely* many prizes.

DEFINITION 2.8. Let X be an infinite set of prizes/consequences, a probability measure p on X is said to be *simple* if it has a *finite support*, that is, if

$$|\text{supp}(p)| = |\{x \in X : p(x) > 0\}| < \infty. \quad (2.18)$$

Denote by \mathcal{L}_X^* the set of all simple probabilities on X , and we refer to \mathcal{L}_X^* as an *extended space of lotteries*. The notational difference between \mathcal{L}_X and \mathcal{L}_X^* is that \mathcal{L}_X contains all the probability measures defined on a *finite* X , whereas, for any $p \in \mathcal{L}_X^*$, p is defined on some *infinite* X but with finite support. Clearly, for any $\lambda \in [0, 1]$ and any simple probabilities p, q , the *mixture* of p and q , written $p \oplus_\lambda q$, is in \mathcal{L}_X^* . And for any $p \in \mathcal{L}_X^*$, p

can be written as the sum of degenerate lotteries that support p , an analogue of (2.2):

$$p = \sum_{x \in \text{supp}(p)} p(x) \delta_x. \quad (2.19)$$

Then a similar argument for Lemma 2.5 leads to the following observation.

LEMMA 2.9. If there exist a \succsim -maximal element x^* and a \succsim -minimal element x_* in X then, for each $p \in \mathcal{L}_X^*$, $\delta_{x^*} \succsim p \succsim \delta_{x_*}$.

THEOREM 2.10. Let \succsim be a preference relation on \mathcal{L}_X^* . Then \succsim satisfies vNM 1-3 if and only if there exists a vNMUF $u \in \mathbb{R}^X$ such that

$$p \succsim q \text{ iff } \sum_{x \in X} p(x)u(x) \geq \sum_{y \in X} q(y)u(y), \quad (2.20)$$

where u is unique up to a positive linear transformation.

PROOF OF THEOREM 2.10. We prove by modifying step (1)-(4) in the proof of Theorem 2.6 with (1*)-(4*) to account for the added assumption that X is infinite and that each $p \in \mathcal{L}_X^*$ is a simple probability measure. We only show the modified steps (1*) and (3*), namely the steps where the assumption of X being infinite plays a role. Steps (2*) and (4*) hold with obvious notational changes.

(1*) If for any $p, q \in \mathcal{L}_X^*$, $p \sim q$, then let u be any constant function, then we are done. Otherwise, fix any p, q satisfying $p \succ q$. By vNM 1, for any $r \in \mathcal{L}_X^*$, exactly one of the following cases holds:

(i) $p \succ r \succ q$, (ii) $r \succ p$, (iii) $q \succ r$.

For case (i), define function $U : \mathcal{L}_X^* \mapsto [0, 1]$ as follows,

$$U(r) := \inf \{ \alpha \in [0, 1] \mid p \oplus_\alpha q \succsim r \}.$$

Then, by Lemma 2.3(4), $r \sim p \oplus_\lambda q$ if and only if $\lambda = U(r)$. It follows that $U(p) = 1$ and $U(q) = 0$. For any r in case (ii), by Lemma 2.3(4), let a be such that $p \sim r \oplus_a q$, define $U(r) = 1/a$. Similarly, for any r in case (iii), let a be such that $q \sim p \oplus_a r$, define $U(r) = a/(a-1)$. Thus, by Lemma 2.3(3), we have that U is a numerical representation of \succsim :

$$p \succsim q \text{ if and only if } U(p) \geq U(q) \text{ for all } p, q \in \mathcal{L}_X^*. \quad (2.21)$$

(3*) Define $u : X \rightarrow \mathbb{R}$ by

$$u(x) = U(\delta_x) \text{ for all } x \in X. \quad (2.22)$$

Then, for any $p, q \in \mathcal{L}_X^*$, a modified step (2*) together with (2.19)-(2.21) yield that

$$\begin{aligned} p \succsim q &\iff U\left(\sum_{x \in \text{supp}(p)} p(x)\delta_x\right) \geq U\left(\sum_{y \in \text{supp}(q)} q(y)\delta_y\right) \\ &\iff \sum_{x \in \text{supp}(p)} p(x)U(\delta_x) \geq \sum_{y \in \text{supp}(q)} q(y)U(\delta_y) \\ &\iff \sum_{x \in X} p(x)u(x) \geq \sum_{y \in X} q(y)u(y). \end{aligned}$$

This completes the proof of the theorem. \square

REMARK 2.11. Note that all the probability functions (either in \mathcal{L}_X or in \mathcal{L}_X^*) considered in this section are simple (have finite support). The theorems proved above hold regardless of whether these probabilities are finitely or countably additive. This ceases to be true if probability functions defined over X are *not* simple: different constraints with different additivity conditions need to be added in order for the representation theorem to hold. See Fishburn (1970, Chapter 8) and Fishburn (1982, Chapter 3) for an extensive discussion for these cases.

3. Horse-race Lotteries

3.1. Risk versus uncertainty. In the von Neumann and Morgenstern expected utility model, the decision maker is uncertain as to which outcome/prize will transpire, where the uncertainty is associated with some objective chances attached to the outcomes. For instance, in a gambling situation, the gambler is uncertain about the outcome of the spin of a roulette wheel, where the betting on each possible outcome comes with a known risk (objective probability), and hence the vNM model is commonly referred to as a decision model under *risk*. Under the assumption of these known objective chances, the vNM expected theory provides a systematic way of retrieving decision makers' subjective utilities of the outcomes given their respective preferences among the probability distributions, i.e., among vNM lotteries. Now, it is conceivable that there are cases where these objective chances might be lacking. Consider, for instance, that in a horse race the gambler needs to choose between two gambles h and h' on the possible outcomes of the horse race: either horse H_1 , or H_2 , or H_3 will be the winning horses and the payoffs of the two gambles are given in the matrix below.

	H_1	H_2	H_3	
h	\$100	0	\$20	
h'	0	\$100	\$20	(3.1)

That is, if gamble h is chosen and horse H_1 wins the race then the gambler will be paid with 100 dollars, 0 if H_2 is the winner, and so on. Here, the winning horses form a set of *possible states of the world*, denoted by S .⁸

As seen, in this example, there are no objective chances involved. In making a decision, the gambler needs to provide his own probabilistic estimation on the winning horse, based perhaps on his knowledge about the horses, past experiences with horse race, or some other considerations. A decision framework that treats this type of decision problems is often referred to as a decision model under *uncertainty*.

A complete treatment of the above case will have to wait until the next Chapter where we present Savage's theory of subjective expected utility. In this section we discuss an *intermediate* step where, instead of receiving direct cash reward, the gambler is paid with some other type of prizes, namely roulette lotteries p, q which indirectly lead to cash reward (e.g., if gamble h is chosen and H_1 wins the horse race then the gambler will be paid with roulette lottery p which in turn says that with 50-50 chance the gambler will get either \$100 or \$20 dollars).

$$\begin{array}{c|ccc} & H_1 & H_2 & H_3 \\ \hline h & p & 0 & q \\ h' & 0 & p & q \end{array} \Rightarrow \begin{array}{c|ccc} & \$100 & \$20 & 0 \\ \hline p & 1/2 & 1/2 & 0 \\ q & 0 & 1/2 & 1/2 \end{array}$$

The distinction between these two types of lotteries, namely horse-race lotteries and roulette lotteries, was made in [Anscombe and Aumann \(1963\)](#), where the roulette lotteries are just vNM lotteries. Their decision model is hence a mixture system containing both subjective and objective probabilities. The goal, as stated by the authors, is "to define the person's probabilities in terms of chances, by an extension of von Neumann-Morgenstern theory."

Let X be a finite set of prizes, \mathcal{L}_X be the lottery space of X , and let S be a finite set of states of the world. A *horse-race lottery* (or *horse lottery* for short) is a function mapping from S to \mathcal{L}_X .⁹ Denote the set of all horse lotteries by \mathcal{H} , that is, $\mathcal{H} = \mathcal{L}_X^S$.

⁸We shall provide an analysis of the nature of the states in our discussion of Savage's decision model later. For the time being, a state of the world is taken as a specification of a possible way that the world may unfold that is relevant to the current decision situation.

⁹In later chapters, we will be discussing the additivity condition of the subjective probability measures derived. This depends on the basic setups of the spaces in which various probability measures are defined. [Anscombe and Aumann \(1963\)](#) were not explicit about the cardinality of the set of prizes on which vNM lotteries are defined. They mentioned in passing that the restriction to finite space in [Luce and Raiffa \(1957\)](#)'s proof of the existence of a vNM utility representation is not necessary, however, the examples they used (i.e., roulette lotteries) and the details of their proofs involved are all finitary in nature. They also compared their system with that of [Savage \(1954\)](#), where the horse lotteries are just a special type of Savage acts (with vNM lotteries as consequences). They did not give further details on the structure of the state space on which their horse lotteries are defined. Here, again, their examples and the proposed axioms (Assumption 1 & 2) all use finite structures. To simplify matters, we discuss the case where all vNM lotteries are simple and the set of states is finite. That is, we consider \mathcal{L}_X instead of \mathcal{L}_X^* .

Given the definition of compound vNM lotteries in (2.3), define the operation of (convex) combination of horse lotteries as follows: a *compound horse lottery* of any $h, h' \in \mathcal{H}$ with scalar $\lambda \in [0, 1]$, in symbols $h \oplus_\lambda h'$, is defined as

$$(h \oplus_\lambda h')(s) =_{\text{Df}} h(s) \oplus_\lambda h'(s) \quad \text{for all } s \in S. \quad (3.2)$$

NOTATION. By definition, $h(s)$ is itself a probability function in \mathcal{L}_X , we often write $h(s)(\cdot)$ as $h_s(\cdot)$ for short. Then, under our current notational convention, h denotes a horse lottery and h_s is a roulette lottery, i.e., a vNM lottery.

Note that, in (3.2), for any given state $s \in S$, it is clear that $h(s) \oplus_\lambda h'(s) \in \mathcal{L}_X$, and hence $h \oplus_\lambda h'$ is also a horse lottery in \mathcal{H} by definition. The task for our decision maker is to choose among the horse lotteries the preferred one(s). The preferences are further represented by a preference relation \succsim , from which her subjective probability measure μ on the occurrences of the states and her subjective utility measure u on the prizes are to be deduced.

3.2. State-dependent utility. Let \succsim be a preference relation (a preorder) on the set of horse lotteries \mathcal{H} . In strict parallel to the von Neumann and Morgenstern postulates vNM 1-3, the first three **Anscombe and Aumann** (A-A) axioms on \succsim take the following form.

A-A 1. \succsim is a complete preorder.

A-A 2. For all $h, h', t \in \mathcal{H}$ and $\lambda \in (0, 1]$,

$$h \succ h' \iff h \oplus_\lambda t \succ h' \oplus_\lambda t.$$

A-A 3. For any $h, h', t \in \mathcal{H}$, there exist $a, b \in (0, 1)$, such that

$$h \succ t \succ h' \implies h \oplus_a h' \succ t \succ h \oplus_b h'.$$

These axioms are sufficient for deriving the following *state-dependent* representation theorem which can be seen as a direct consequence of Corollary 2.7.

THEOREM 3.1. Let \succsim be a preference relation on $\mathcal{H}_{S,X}$, then \succsim satisfies A-A 1-3 if and only if there exist (state-dependent utility) functions $u : S \times X \mapsto \mathbb{R}$ such that, for any $h, h' \in \mathcal{H}_{S,X}$,

$$h \succsim h' \text{ iff } \sum_{s \in S} \sum_{x \in X} h_s(x) u(s, x) \geq \sum_{s \in S} \sum_{x \in X} h'_s(x) u(s, x). \quad (3.3)$$

The right hand side of (3.3) takes the full advantage of the fact that each h_s is itself a probability function, the task is then to show that there exists a utility function such that the comparison between horse lotteries can be represented by their expected utilities.

PROOF. We give only the non-trivial “only if” direction of the proof. Let $\mathcal{L}_{S \times X}$ denote the set of probability functions defined on $S \times X$. Note that, for each horse lottery $h \in \mathcal{H} = \mathcal{L}_X^S$, there corresponds a $\hat{h} \in \mathcal{L}_{S \times X}$ such that for any $s \in S$ and any $x \in X$, $h_s(x) = |S| \cdot \hat{h}(s, x)$. Let $\hat{\mathcal{H}}$ be the set of all \hat{h} 's, that is,

$$\hat{\mathcal{H}} = \left\{ \hat{h} \mid h_s(x) = |S| \cdot \hat{h}(s, x) \text{ and } h \in \mathcal{H} \right\}. \quad (3.4)$$

Thence $\hat{\mathcal{H}}$ is subset of $\mathcal{L}_{S \times X}$ which is convex and compact (because \mathcal{H} is). Further, define an ordering $\hat{\succsim}$ on $\hat{\mathcal{H}}$ such that, for any $\hat{h}, \hat{h}' \in \hat{\mathcal{H}}$,

$$\hat{h} \hat{\succsim} \hat{h}' \iff h \succsim h'. \quad (3.5)$$

It is easy to see that \succsim on \mathcal{H} satisfies A-A 1-3 if and only if the induced ordering $\hat{\succsim}$ on $\hat{\mathcal{H}}$ satisfies vNM 1-3. By compactness and vNM 3 (i.e., continuity), there exists an $\hat{\succsim}$ -maximum and an $\hat{\succsim}$ -minimum in $\hat{\mathcal{H}}$. Hence, by Corollary 2.7 and (3.5), there exists a vNMUF $v \in \mathbb{R}^{S \times X}$ such that

$$\begin{aligned} h \succsim h' &\iff \sum_{(s,x) \in S \times X} \hat{h}(s,x)v(s,x) \geq \sum_{(s,x) \in S \times X} \hat{h}'(s,x)v(s,x) \\ &\iff \sum_{s \in S} \sum_{x \in X} \frac{1}{|S|} h_s(x)v(s,x) \geq \sum_{s \in S} \sum_{x \in X} \frac{1}{|S|} h'_s(x)v(s,x). \end{aligned}$$

The proof is completed once we define $u(s, x)$ to be $v(s, x) / |S|$. \square

As seen, the derived two-place utility function u is *state-dependent* as the function value also depends on the state. For any $s \in S$, we write $u(s, \cdot)$ as $u_s(\cdot)$ and refer to the latter as the utility function with respect to state s . Theorem 3.1 then states that the agent's preference relation among horse lotteries can be represented using a series of state-dependent utility functions $\{u_s\}_{s \in S}$.

3.3. State-independent utility. Theorem 3.1 can be further strengthened by adding one more axiom so that the representation takes the form of a combination of agent's subjective probability on states and her subjective *state-independent* utility on the prizes. The strengthening relies on the following concept of “constant horse lotteries”.¹⁰

DEFINITION 3.2. A horse lottery is said to be *constant* with respect to $p \in \mathcal{L}_X$, written \mathfrak{c}_p if $\mathfrak{c}_p(s) = p$ for all $s \in S$.

It is clear from the definition above that each constant horse lottery can be identified with a vNM lottery. This then enables us to define a preference ordering $\hat{\succsim}^*$ on \mathcal{L}_X using the preference relation \succsim over \mathcal{H} as follows

$$p \hat{\succsim}^* q \iff \mathfrak{c}_p \succsim \mathfrak{c}_q \quad \text{for all } p, q \in \mathcal{L}_X. \quad (3.6)$$

¹⁰Constant horse lotteries are special cases of Savage's notion of “constant acts” in Definition 5.3 below. More discussion on Savage's constant acts will be given in Section ??.

Call \succ^* the preference relation on \mathcal{L}_X induced by \succ .

LEMMA 3.3. Let \succ be a preference on \mathcal{H} and \succ^* the corresponding induced preference on \mathcal{L}_X , if \succ satisfies A-A 1-3 then \succ^* satisfies vNM 1-3.

PROOF. We prove the lemma by direct verifications.

- (1) vNM 1 can be easily verified using the definition of constant horse lottery.
 (2) For vNM 2, suppose that $p \succ^* q$, then we have $c_p \succ c_q$ via (3.6). By A-A 2, for any $r \in \mathcal{L}_X$ and $\lambda \in (0, 1]$,

$$c_p \succ c_q \iff c_p \oplus_\lambda c_r \succ c_q \oplus_\lambda c_r.$$

By (3.2), $c_p \oplus_\lambda c_r(s) = c_p(s) \oplus_\lambda c_r(s)$ for all $s \in S$. Since c_p, c_q, c_r are constant horse lotteries, we have that $c_p(s) \oplus_\lambda c_r(s) = p \oplus_\lambda r$ for all $s \in S$. Similarly, $c_q \oplus_\lambda c_r = q \oplus_\lambda r$. Hence $c_p \oplus_\lambda c_r \succ c_q \oplus_\lambda c_r$ if and only if $c_{p \oplus_\lambda r} \succ c_{q \oplus_\lambda r}$, thence $p \oplus_\lambda r \succ^* q \oplus_\lambda r$ via (3.6).

- (3) Finally, suppose that $p \succ^* r \succ^* q$, then by (3.6) we have $c_p \succ c_r \succ c_q$. By A-A 3, there exists $a, b \in (0, 1]$ such that $c_p \oplus_a c_q \succ c_r \succ c_p \oplus_b c_q$. Using a similar argument as in (2), we get $c_{p \oplus_a q} \succ c_r \succ c_{p \oplus_b q}$. Therefore, $p \oplus_a q \succ^* r \succ^* p \oplus_b q$ via (3.6), and hence vNM 3. \square

Further, a state $s \in S$ is said to be *null* if the agent is indifferent between any horse lotteries that differ only on s , s is *non-null* if it is not null. We are now in the position to state the fourth A-A axiom which facilitates state-independent utility representation.

A-A 4. For any $h, h' \in \mathcal{H}$,

- (1) if $h_s \succ^* h'_s$ for all $s \in S$ then $h \succ h'$;
 (2) if $h_s \succ^* h'_s$ for all $s \in S$ and $h_s \succ^* h'_s$ for some non-null $s \in S$ then $h \succ h'$;

where \succ^* is the preference on \mathcal{L}_X induced by \succ .

Axiom A-A 4 is commonly known as the *Monotonicity* axiom (or sometimes the *Dominance* or *State-independent* axiom). It asserts that horse lottery h weakly dominates h' if, in each state s , the vNM lottery h_s weakly dominates h'_s (under the induced preference ordering through the notion of constant horse lottery); h strictly dominates h' if, for some state s , h_s strictly dominates h'_s . As shown the in following lemma, the axiom regulates in a very rigid way the two preferential systems (\succ and \succ^*).

LEMMA 3.4. Let \succ and \succ^* be as above. Suppose that \succ satisfies A-A 1-4 and that, for any $s \in S$, u_s be a utility function obtained in Theorem 3.1, then u_s is a vNMUF with respect to \succ^* .

PROOF. By Lemma 3.3, \succsim^* satisfies vNM 1-3, then it suffices to show that, for any $p, q \in \mathcal{L}_X$,

$$p \succsim^* q \text{ if and only if } \sum_{x \in X} p(x)u_s(x) \geq \sum_{x \in X} q(x)u_s(x).$$

From $p \succsim^* q$ we have $c_p \succsim c_q$ by definition. Fix s , let h be the horse lottery that differs from c_q in precisely the following way

$$h(v) = \begin{cases} p & \text{if } v = s \\ q & \text{if } v \neq s \end{cases}.$$

That is, h yields p at s but agrees with c_q at all other states. Then, by A-A 4, we have $p \succsim^* q$ iff $h \succsim c_q$. It follows, by (3.3), that

$$\begin{aligned} h \succsim c_q &\Leftrightarrow \sum_{v \in S} \sum_{x \in X} h_v(x)u_v(x) \geq \sum_{v \in S} \sum_{x \in X} q(x)u_v(x) \\ &\Leftrightarrow \sum_{x \in X} p(x)u_s(x) + \sum_{v \in S \setminus \{s\}} \sum_{x \in X} q(x)u_v(x) \geq \sum_{v \in S} \sum_{x \in X} q(x)u_v(x) \\ &\Leftrightarrow \sum_{x \in X} p(x)u_s(x) \geq \sum_{x \in X} q(x)u_s(x). \end{aligned}$$

This completes the proof of the lemma. \square

THEOREM 3.5 (Anscombe and Aumann). Let \succsim be a preference relation on \mathcal{H} . Then \succsim satisfies A-A 1-4 if and only if there exists a utility function $u \in \mathbb{R}^X$ and a probability measure μ on S such that, for any $h, h' \in \mathcal{H}$,

$$h \succsim h' \text{ iff } \sum_{s \in S} \mu(s) \sum_{x \in X} h_s(x)u(x) \geq \sum_{s \in S} \mu(s) \sum_{x \in X} h'_s(x)u(x). \quad (3.7)$$

PROOF. By Theorem 3.1, there exists a series of state-dependent functions $\{u_s\}_{s \in S}$ such that (3.3) holds. Further, Lemma 3.4 shows that the u_s 's are vNM utility representations with respect to the same preference relation \succsim^* over lotteries, and hence are unique up to positive affine transformations. That is, if we fix a state s' and let $u = u_{s'}$, then for any $s \in S$, there, by (2.12), exist a_s, b_s ($a_s > 0$) such that $u_s = a_s u + b_s$. Then, from (3.3), we get

$$h \succsim h' \text{ iff } \sum_{s \in S} a_s \sum_{x \in X} h_s(x)u(x) \geq \sum_{s \in S} a_s \sum_{x \in X} h'_s(x)u(x). \quad (3.8)$$

Now, define $\mu : S \mapsto \mathbb{R}_+$ to be such that

$$\mu(s) = \frac{a_s}{\sum_{v \in S} a_v}. \quad (3.9)$$

This, together with (3.8), yield what we want. \square

In the A-A system above, (3.9) is interpreted as the agent's subjective probability, which, as seen, is defined in terms of the coefficients of a series of vNM utility functions

which, in turn, are defined through vNM lotteries. The model is hence a dualistic system featuring both subjective and objective probabilities. In the next chapter, we introduce Savage's theory of expected utility where probabilities are given a purely subjective and decision-theoretic interpretation.

CHAPTER II

Savage's Subjectivism

Personalistic views hold that probability measures the confidence that a particular individual has in the truth of a particular proposition, for example, the proposition that it will rain tomorrow. These views postulate that the individual concerned is in some ways "reasonable," but they do not deny the possibility that two reasonable individuals faced with the same evidence may have different degrees of confidence in the truth of the same proposition.

— Savage (1972)

4. Introduction

This chapter introduces Leonard J. Savage's theory of subjective expected utility as presented in his seminal book *the Foundations of Statistics*.¹ As indicated in the opening quote, one main objective of this project is to provide a subjective interpretation of the central notion employed in virtually all stages of statistical inferences, namely the notion of probability. Built on earlier works of Frank Ramsey, Bruno de Finetti, John von Neumann and Oskar Morgenstern, among others, Savage's theory seeks to ground a theory of personal probability in a normative theory of rational decision making of highly idealized reasonable agents, where by "reasonable agents" Savage means individuals who are capable of distinguishing "between coherent behavior and blunder, or demonstrable incoherence, in the face of uncertainty." This is achieved by prescribing various rationality principles and structural assumptions governing decision makers' behaviors in decision-making situations, by which the agents can police their own potential decisions against incoherency.

Savage's theory begins with the decision maker's preferences over her potential actions, modeled by a binary preference relation. A set of axioms is postulated on this preference relation. From the first five postulates a comparative notion of subjective probability is derived which reflects the agent's qualitative probabilistic judgments over possible circumstances under which these actions are taking place. With the sixth postulate, the derived qualitative probability is further represented by a numerical probability measure together with a personal utility function for simple acts (i.e., acts that may lead

¹The first edition of Savage's book where the axiomatic theory was first introduced appeared in 1954 published by John Wiley & Sons. All citations in this dissertation refer to the second revised edition published by Dover Publications in 1972.

to finitely many potential consequences under different states). The last postulate is brought in so that the utility function for simple acts can be extended to all acts (cf. Table 4.1).

TABLE 4.1. Inferential order in Savage's system.

P1-5		+ P6		+ P7
Qualitative probability	\Rightarrow	Quantitative probability Utility for simple acts	\Rightarrow	Utility for all acts

Savage's approach differs from the methods adopted by Ramsey and Anscombe-Aumann in that, in the latter cases, the agents' subjective probabilities are derived from their personal utilities, which in turn are constructed based on some presupposed chance mechanisms (or, in the case of Ramsey, the notion of ethically neutral propositions, which plays a similar role as an unbiased coin receiving objective probability $1/2$). This inferential order is reversed in Savage's subjectivism where the preference relation over acts is taken as the only primitive notion, from which the agent's personal probabilities and utilities are subsequently revealed. As a result of this methodological reversal, Savage's approach may appear to have some computational disadvantages in the sense that the mathematical representation theorem given by Savage is considerably more involved than many of its alternatives, yet the theory is conceptually significant in that the system is maintained as a purely subjective framework with no direct reference to objective probabilities.

Our expositions will follow closely Savage's original approach. The plan is as follows. After an introduction of basic definitions and notations in Section 5.1, we provide an analysis of the well-known "sure-thing" principle (Section 5.2). This will be followed by a reconstruction of Savage's theory of qualitative probability (Section 6.1), quantitative probability (Section 6.2), and personal utility for simple acts (Section 7.1). In Section 7.2, we investigate the role of Savage's last postulate (i.e., P7) played in extending utility from simple acts to general acts.

5. Decision Matrix

5.1. States, consequences, and acts. The basic setup of Savage's decision model can be illustrated in the decision matrix in Table 5.1, where $S = \{s_1, s_2, \dots\}$ is an (infinite) set of *states of the world* specifying those possible circumstances that are relevant to the decision situation at hand,² $X = \{o_{1,1}, o_{1,2}, \dots\}$ is a (finite or infinite) set of *consequences* (or *outcomes*), and f_1, f_2, \dots are commonly referred to as (Savage) *acts*, which are arbitrary

² In fact, as a feature of Savage's theory, S must contain uncountably many states, we will return to this point later (cf. Remark 6.16 below).

functions mapping from S to X . The intended interpretation of an act f_m is that the agent's choice of f_m will lead to consequence $o_{m,n}$ if s_n is the true state of the world. Denote the set of all acts by \mathcal{A} .

TABLE 5.1. Savage's decision matrix.

	s_1	s_2	\cdots	s_n	\cdots
f_1	$o_{1,1}$	$o_{1,2}$	\cdots	$o_{1,n}$	\cdots
f_2	$o_{2,1}$	$o_{2,2}$	\cdots	$o_{2,n}$	\cdots
\vdots			\ddots		
f_m	$o_{m,1}$	$o_{m,2}$	\cdots	$o_{m,n}$	\cdots

As a primitive assumption, the agent is assumed to have *preferences* over acts, which are modeled by a preorder \succsim on \mathcal{A} . Thus, for any acts $f, g \in \mathcal{A}$, $f \succsim g$ is taken to mean that act f is *weakly preferred* to act g (or that g is not preferred to f) by the agent. We define $f \succ g \stackrel{\text{Df}}{=} f \succsim g$ and $g \not\succeq f$. This means that f is strictly preferred to g . And define $f \sim g \stackrel{\text{Df}}{=} f \succsim g$ and $g \succsim f$, that is, f and g are *equi-preferable* (or that f is *indifferent* to g).

DEFINITION 5.1 (Combined acts). For any $f, g \in \mathcal{A}$, define the *combination* of f and g with respect to an event E (a set of states), written $f \oplus_E g$, to be such that:

$$(f \oplus_E g)(s) = \begin{cases} f(s) & \text{if } s \in E \\ g(s) & \text{if } s \in E^C, \end{cases} \quad (5.1)$$

where $E^C = S - E$ is the compliment of E .³

That is to say, $f \oplus_E g$ is the act which agrees with f on event E , with g on E^C , and it is easily seen that $f \oplus_E g \in \mathcal{A}$. Using the concept defined in (5.1), we can interpret $(f \oplus_E g) \oplus_F h$ as saying: do f if $E \cap F$ obtains, g if $F \cap E^C$ occurs, and h if F^C , and so on. The following is a list of simple properties of operation \oplus_E . The proof is immediate from the definition and omitted.

LEMMA 5.2. For any $E, F \in \mathcal{F}$, and for any acts $f, g \in \mathcal{A}$,

- (1) $f \oplus_E g = g \oplus_{E^C} f$;
- (2) $(f \oplus_E g) \oplus_F h = f \oplus_{E \cap F} h$;
- (3) $f \oplus_E (f \oplus_F g) = f \oplus_{E \cup F} g$;
- (4) $(f \oplus_E g) \oplus_{E^C} h = h$.

The following concept is a key structural component of Savage's theory, which will play an important role in the discussions that follows.

³Some writers use ' (f, E, g) ' or ' fEg ' or ' $f|E + g|E^C$ ' or ' $[f \text{ on } E, g \text{ on } E^C]$ ' for combined acts.

DEFINITION 5.3 (Constant acts). For any $a \in X$, an act is said to be *constant* with respect to consequence a , in symbols c_a , if

$$c_a(s) = a \quad \text{for all } s \in S. \quad (5.2)$$

In other words, act c_a “constantly” outputs consequence a no matter which state $s \in S$ transpires. Now, given a preference ordering \succsim on \mathcal{A} , an ordering \succsim^* over consequences can be defined using constant acts by

$$a \succsim^* b \iff c_a \succsim c_b \quad \text{for all } a, b \in X. \quad (5.3)$$

That is to say, consequence a is said to be weakly preferred to consequence b if the constant act c_a is weakly preferred to c_b . Call \succsim^* the preference relation on X induced by \succsim . For notational purpose, we often use the same symbol ‘ \succsim ’ for both the preference relation over acts and the induced preference over consequences and let the context determine on which set of alternatives a given preference \succsim is defined. With these two preference orderings, we proceed to define a (qualitative) relation among events.

DEFINITION 5.4. For any events $E, F \in \mathcal{F}$, say that E is *weakly more probable* than F , written $E \succeq F$ (or $F \preceq E$), if, for any $a, b \in X$ with $a \succsim b$,

$$c_a \oplus_E c_b \succsim c_a \oplus_F c_b \quad (5.4)$$

(or equivalently if $c_b \oplus_F c_a \succsim c_b \oplus_E c_a$). E and F are said to be *equally probable*, in symbols $E \simeq F$, if both $E \succeq F$ and $F \succeq E$ hold.

The definition says that the agent’s belief that E is more probable than F is manifested in her preference for the compound act $c_a \oplus_E c_b$ which, in turn, is determined by the agent’s subjective estimation of the likelihood of obtaining the more favorable constant act c_a . (A postulate, i.e. SVG 4, will be inserted in order to ensure that the notion of one event being more probable than another is well defined, that is, the definition in (5.4) does not depend on the choice of a, b .)

REMARK. 1. Savage’s “simple ordering” is, in our terminology, a total preorder. He uses ‘ F ’ for the set of consequences and he characterizes total preorders as “simple orderings”. In particular, he uses boldface letters $\mathbf{f}, \mathbf{g}, \dots$ for acts and italics f, g, \dots for values of “acts that are constant”, writing $\mathbf{f} \equiv g$ when $\mathbf{f}(s) = g$ for all states s . He also uses ‘ f ’ for constant act whose value is f . Furthermore, he sometimes switches to italicized notation even when the function is not constant, as he does in the statement of P4 on p.31, where he writes $f_A(s)$ instead of $\mathbf{f}_A(s)$, or in Theorem 1 on page 70, where he writes $f(s) = f_i$ instead of $\mathbf{f}(s) = f_i$ as he should.

2. As seen from the definitions above, the construction of constant acts c_a ($a \in X$) plays a central role in associating various concepts in Savage’s decision model, and it is

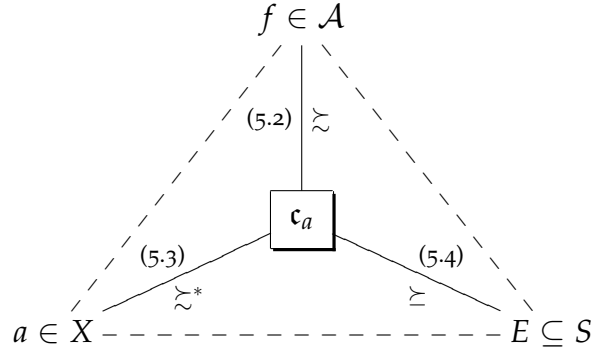


FIGURE 5.1. Constant act c_a and other parameters in Savage's model.

through the notion of constant acts that different binary relations are interrelated (see Figure 5.1). This notion, however, is highly problematic. We address the issues brought by the assumption of the existence of constant acts (one for each consequence) in great detail in [Gaifman and Liu \(2015\)](#) where we provide a simplification of Savage's theory without appealing to constant acts. The exposition in this chapter however will still use constant acts.

The goal is to show that the defined "more probable" relation \succeq is a qualitative probability (will be made precise below) and that there exists a unique numerical probability measure μ on (S, \mathcal{F}) such that⁴

$$E \succeq F \iff \mu(E) \geq \mu(F), \quad \text{for all } E, F \in \mathcal{F}, \quad (5.5)$$

and that there is a real-valued function u on X for which

$$f \succsim g \iff \int_S u[f(s)] \mu(ds) \geq \int_S u[g(s)] \mu(ds), \quad (5.6)$$

for all $f, g \in \mathcal{A}$, where u is unique up to a positive transformation. This is Savage's main *representation theorem* we seek to prove, which will be discussed in Section 6 and Section 7. But before proceeding to detailed reconstruction, let us first analyze Savage's well-known "sure-thing" principle and its formal configurations.

5.2. The sure-thing principle and postulate 2. The cornerstone of Savage's decision model is a postulated rationality principle known as the "sure-thing principle". The following is the example used by Savage to motivate this principle.

⁴Savage stated explicitly that in his theory probability is defined for all events which are taken to be all subsets of S , and hence $\mathcal{A} = X^S$ ([Savage, 1972](#), p.40). For our purpose, we restrict our attention to "measurable acts." That is to say, given measurable spaces (S, \mathcal{F}) and (X, \mathcal{G}) where \mathcal{F} and \mathcal{G} are some σ -algebras equipped on S and X , respectively, we consider only those functions (acts) that are measurable \mathcal{F}/\mathcal{G} .

EXAMPLE 5.5 (Businessman). A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness the purchase. So, to clarify the matter for himself, he asks whether he would buy if he knew that the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew that the Democratic candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. \triangleleft

As illustrated in this example, the decision-theoretic principle under consideration stems from an intuitive idea of reasoning by cases that if a decision maker takes certain course of action given the occurrence of some event and she will do the same if the event does *not* occur, then she shall proceed with *that* action without taking into account as to whether or not the event takes place, in other words, the implementation of the course of action is a “sure-thing”. To state in terms of preferences over acts, the sure-thing principle says that

STP: If the decision maker prefers one act over another assuming either certain event obtains or that the compliment of the event obtains, then her preference over the two acts shall remain unchanged.

The principle is sometimes referred to as the *dominance principle*, which can be stated more generally as follows:

STP_n: if the state space is partitioned into n -many mutually exclusive cells, which represent n different decision situations, and if the consequence of one act weakly dominates that of another in each one of these possible situations, then the act is weakly preferred throughout.

Savage takes this consideration to be fundamental to rational decision making: “I,” he says, “know of no other extra-logical principle governing decisions that finds such ready acceptance” (*ibid.* p.21).

5.2.1. *Conditional preference.* Note that the statement of the sure-thing principle above employs explicitly a concept of conditional preference, that is, one act being preferred to another *given* the occurrence of certain event. Since the current formal setup is based entirely on *unconditional* preferences over acts, the notion of conditional preference is not directly expressible. Some alternative arrangements hence need to be made.

DEFINITION 5.6 (Conditional preference). Let E be some event, then, given acts $f, g \in \mathcal{A}$, f is said to be weakly preferred to g *given* E , written $f \succsim_E g$, if, for *all* pairs of acts $f', g' \in \mathcal{A}$, we have

- (1) f (g) agrees with f' and g agrees with g' on E ,

TABLE 5.2. Illustrations of

(A) Conditional preference

	E	E^C
f	a	c
g	b	d
f'	a	e
g'	b	e

(B) Savage's postulate 2

	E	E^C
f	a	c
g	b	c
f'	a	d
g'	b	d

(2) f' agrees with g' on E^C , and(3) $f' \succsim g'$.

That is,

$$\left. \begin{array}{l} f(s) = f'(s), g(s) = g'(s) \quad \text{if } s \in E \\ f'(s) = g'(s) \quad \quad \quad \quad \quad \text{if } s \in E^C. \end{array} \right\} \implies f' \succsim g'. \quad (5.7)$$

In other words, the conditional preference of f over g on the occurrence of event E is defined in terms of all unconditional preferences of f' over g' under the constraints that f' and g' agree, respectively, with f and g on event E and with each other on E^C , and that f' unconditionally weakly preferred to g' . Table 5.2a contains an illustration of conditional preference, where $\{E, E^C\}$ forms a simple partition of S for which $f(s) = a$ for all $s \in E$ and $f(s) = c$ for $s \in E^C$ (other acts defined similarly). Then the definition of conditional preference says that f is weakly preferred to g given E if $f' \succsim g'$ for all such f' s and g' s. Now, given the definition of conditional preference, **STP** can be translated into⁵

$$\left[f \succsim_E g, f \succsim_{E^C} g \right] \implies f \succsim g. \quad (\text{STP})$$

Savage, however, was unwilling to incorporate (STP) directly into his system for the reason that the concept of conditional preference is based on knowledge of the possible occurrences of some events, the introduction of which may lead to, it is said, unsought philosophical complications.⁶ Instead, he posited a different principle which is a technical approximation to **STP** known as the *formal* version of the sure-thing principle, and he left **STP** itself as an *informal* or, to use his phrase, a “loose” version of the sure-thing principle.

⁵In what follows, we use the boldface **STP** to refer to the informal statement of the principle and use (STP) to refer to its formulation in the formal model, same for **P2** and (P2) below.

⁶Savage (1972, p.22) explains: “The sure-thing principle [i.e., **STP** above] cannot appropriately be accepted as a postulate in the sense that P1 is, because it would introduce new undefined technical terms referring to knowledge and possibility that would refer it mathematically useless without still more postulates governing these terms.” See Gaifman (2013, p.375) for a critique of this line of argument, where it is pointed out that Savage is guilty of confusing hypothetical reasoning with counterfactual knowledge: it is the former, not the latter, that is involved in formulating the sure-thing principle, which is conceptually transparent and non-problematic.

This alternative principle contains no direct reference to conditional preferences and is officially stated as his second postulate (P2) for rational decision making, which says that, for any acts f, g, h, h' and for any event E ,

$$f \oplus_E h \succsim g \oplus_E h \iff f \oplus_E h' \succsim g \oplus_E h', \quad (\text{P2})$$

As the example in Table 5.2b illustrates, if f' and g' agree, respectively, with f and g on E and with each other on E^C , then (P2) mandates $f \succsim g$ iff $f' \succsim g'$. Here we remark that one technical motivation for imposing (P2) is to provide a provision to the definition of conditional preference in Definition 5.6 so that the notion is well defined. (Notice that, in the absence of (P2), an act f may fail to be conditionally preferred to another act g (i.e. $f \not\succeq_E g$) if there exist two pairs of acts (f', g') and (f'', g'') satisfying both conditions (1) and (2) for which $f' \succsim g'$ and $f'' \not\succeq g''$. This possibility for f to fail to be conditionally preferred to g is excluded by (P2), under which $f \not\succeq_E g$ if and only if, for all f' and g' satisfying (1) and (2), $f' \not\succeq g'$.) Beyond this technical reason for invoking (P2) as an additional constraint on the notion of conditional preferences, the rationale behind (P2) as a self-standing rationality principle can be characterized as follows

P2: If the consequences of two acts differ on the occurrence of some event E but otherwise agree with each other, then their preferential comparison between these two acts shall be decided on those states in E and their corresponding consequences.

What underlies this principle seems to be the simple consideration that the difference between any two items is distinguished by the parts where they are actually different. Then P2 implies that if two sets of decision problems have identical payoff structures on E but otherwise have respectively in-differentiable payoffs on E^C then if an option is preferred in the first set of decision problem it should also be favored in the second. Yet, we stress that, even with the presence of this compelling intuition, P2 is after all a different and, in fact, more restrictive principle than STP. We illustrate this point by showing that (P2) is strictly stronger than (STP) in the current formal model.⁷

LEMMA 5.7. Let \succsim be a preorder on \mathcal{A} , then (P2) implies (STP).

PROOF. Assuming (P2), it is easily seen that the definition of conditional preference can be equivalently stated as follows

$$f \succsim_E g \iff f \oplus_E h \succsim g \oplus_E h, \text{ for all } h \in \mathcal{A}. \quad (5.8)$$

⁷Gaifman (2013, p.376) outlined a general method of distinguishing STP from P2 in a partial-act based system, where a partial-act is a (partial) function defined on some event and maybe undefined on other events. And it was shown that the counterpart of P2 in a partial-act system is *independent of* that of STP. Here, we point out that, as far as Savage's total-act system is concerned, (P2) does imply (STP), but not vice versa.

Then the left-hand side of (STP) yields, via (5.8), that

$$f \oplus_E h \succsim g \oplus_E h \quad (5.9)$$

$$f \oplus_{EC} h' \succsim g \oplus_{EC} h'. \quad (5.10)$$

where h and h' are arbitrary acts in \mathcal{A} . Now, in (5.9), substitute h with $h \oplus_E f$, then, by (P2), we get $f = f \oplus_E (h \oplus_E f) \succsim g \oplus_E (h \oplus_E f) = g \oplus_E f$. Similarly, in (5.10), replace h' with $g \oplus_E h'$, then $f \oplus_{EC} g = f \oplus_{EC} (g \oplus_E h') \succsim g \oplus_{EC} (g \oplus_E h') = g$. Together we have that $f \succsim g \oplus_E f$ and $f \oplus_{EC} g \succsim g$. Note that, by Lemma 5.2(1), $g \oplus_E f = f \oplus_{EC} g$, therefore, by transitivity of \succsim , we have $f \succsim g$. \square

The converse, however, does not necessarily hold, that is, there are situations in which (STP) is satisfied but (P2) is violated as shown in the following example.

EXAMPLE 5.8. Let $S = \{s_1, s_2\}$, $X = \{a, b\}$. Then there are four possible acts mapping from S to X as illustrated in the table below.⁸

	s_1	s_2
f_1	a	a
f_2	b	a
f_3	a	b
f_4	b	b

Consider the case where $f_1 \succsim f_2 \sim f_3 \prec f_4$. Then it is easy to see that P2 is violated but (STP) is trivially satisfied. (This is because our example is so arranged that, for any acts $f, g \in \{f_1, f_2, f_3, f_4\}$, (1) if f is different from g then, at least one of the conditional preferences $f \succsim_{s_1} g$ and $f \succsim_{s_2} g$ fails,⁹ in which case the antecedent of (STP) is false, and hence the conditional true; (2) if f is identical to g then (STP) is trivially true). \triangleleft

Lemma 5.7 shows that Savage's proposed (P2) is deductively sufficient for enforcing (STP), however, as shown in Example 5.8, it is more demanding than what (STP) is intended for. Let us summarize the above discussion in the following theorem.

THEOREM 5.9. Let \succsim be a complete preorder on \mathcal{A} , then

- (1) (P2) \implies (STP),
- (2) (STP) $\not\implies$ (P2).

To be sure, the reason that (P2) and (STP) are not deductively equivalent in Savage's system is largely due to the peculiar way how conditional preferences are formulated in his model, where the concept of conditional preference and (P2) are essentially interlocked. Gaifman (2013) suggested a way of defining conditional preference in a more

⁸Strictly speaking, the state space S in Savage system needs to contain uncountably many states (cf. Footnote 2). In writing $S = \{s_1, s_2\}$ we can assume that S is partitioned into two events s_1 and s_2 .

⁹We write $f \succsim_{\{s_1\}} g$ as $f \succsim_{s_1} g$ for short, same below.

straight forward manner so that **STP** can be formulated directly without going through Savage's roundabout way of using mutually dependent notions of conditional preference and **(P2)**. Our discussions and generalizations in later sections will still be made within Savage's framework with total-acts, we, however, emphasize on a clear distinction between **STP** and **P2**, and their formalizations.

5.2.2. *Null events.* Further, an event $E \subseteq S$ is said to be a *null event* if, for any $f, g \in \mathcal{A}$, $f \succsim_E g$, that is, the agent is indifferent between any two acts *given* the occurrence of E . Intuitively, null events are those events whose occurrences take no effect in the agent's decision procedure as the individual believes that it is impossible that they obtain. As we shall soon see, in the current system null events corresponds to those events that receive probability zero. The following is a list of basic properties of null events.

LEMMA 5.10. Let E be a null event, then given **P2**,

- (1) $E \simeq \emptyset$;
- (2) if $f(s) = f'(s)$ and $g(s) = g'(s)$ for all $s \in E^C$, then $f \succsim g$ iff $f' \succsim g'$;
- (3) if $f(s) = g(s)$ for all $s \in E^C$, then $f \sim g$.

PROOF. (1) Let $a, b \in X$ be such that $a \succ b$. Since E is null, we have $c_a \succsim_E c_b$. This implies, by **(P2)** and (5.8), that

$$c_a \oplus_E c_b \succsim c_b \oplus_E c_b = c_b = c_b \oplus_{\emptyset} c_b.$$

By Definition 5.4, $E \succeq \emptyset$. Similarly, from E being null we get $c_b \succsim_E c_a$, thence

$$c_a \oplus_{\emptyset} c_b = c_b \oplus_E c_b \succsim c_a \oplus_E c_b.$$

By definition, $\emptyset \succeq E$. Together, we have $E \simeq \emptyset$.

- (2) By symmetry, we show $f \succsim g$ implies $f' \succsim g'$. Note that, since E is null, we have $f' \succsim_E g'$. Then by **(STP)**, we only need to show that $f' \succsim_{E^C} g'$. By the definition of conditional preference and **(P2)**, it's sufficient to show that, there exists some $h \in \mathcal{A}$ such that

$$f' \oplus_{E^C} h \succsim g' \oplus_{E^C} h. \quad (5.11)$$

Since f' and g' agree respectively with f and g on E^C , (5.11) holds iff

$$f \oplus_{E^C} h \succsim g \oplus_{E^C} h.$$

Take h to be f , then the proof is completed if it can be shown that

$$f \succsim f \oplus_E g. \quad (5.12)$$

To this end, note that since E is null, we have $g \succsim_E f \oplus_E g$, it follows, through (5.8), that, for any $t \in \mathcal{A}$, $g \oplus_E t \succsim (f \oplus_E g) \oplus_E t$. Let $t = g$, then this together with the assumption $f \succsim g$ yield (5.12), which is what we want.

(3) This is an easy consequence of (2). \square

Lemma 5.10(2) says that given any pairs of acts, if they differ pair-wisely only on events that are considered null then their relative preferences will remain the same (cf. the table below).

TABLE 5.3. $f \succsim g$ iff $f' \succsim g'$

	E (null)	E^C
f	a	e
g	b	f
f'	c	e
g'	d	f

As we shall see, this property plays an important role in deriving a utility function for consequences.

6. Subjective Probability

6.1. Qualitative probability. As the first step of our reconstruction of Savage's expected utility representation theory, we introduce the following concept of qualitative probability:

DEFINITION 6.1 (Qualitative probability). Let S be a nonempty set, a binary relation \succsim on S is said to be a *qualitative probability* if, for any $A, B, C \in \mathcal{F}$,

- i. \succsim is a weak order (reflexive, transitive, and complete),
- ii. $A \succ \emptyset$,
- iii. $S \succ \emptyset$,
- iv. $A \succ B$ if and only if $A \cup C \succ B \cup C$, provided $A \cap C = B \cap C = \emptyset$.

where \succ is the strict (i.e., the asymmetric) part of \succsim .

We show that if the preference relation \succsim over acts satisfies the following list of axioms postulated by Savage then the binary relation \succeq over events (sets of states) defined in (5.4) is a qualitative probability.¹⁰

SVG 1. \succsim is a weak order (complete preorder).

SVG 2. For any $f, g \in \mathcal{A}$ and for any $E \subseteq S$, $f \succsim_E g$ or $g \succsim_E f$.

SVG 3. For any $a, b \in X$ and for any non-null event $E \subseteq S$, $c_a \succsim_E c_b$ if and only if $a \succsim b$.

¹⁰SVG 1-5 correspond respectively to P1-5 in Savage (1972), the only difference is that we present these postulates using the notations adopted here, same for SVG 6 and SVG 7 below.

SVG 4. For any $a, b, c, d \in X$ satisfying $a \succsim b$ and $c \succsim d$ and for any events $E, F \subseteq S$, $\mathbf{c}_a \oplus_E \mathbf{c}_b \succsim \mathbf{c}_a \oplus_F \mathbf{c}_b$ if and only if $\mathbf{c}_c \oplus_E \mathbf{c}_d \succsim \mathbf{c}_c \oplus_F \mathbf{c}_d$.

SVG 5. For some constant acts $\mathbf{c}_a, \mathbf{c}_b \in \mathcal{A}$, $\mathbf{c}_b \succ \mathbf{c}_a$.

Here, SVG 1 says that the preference relation is reflexive, transitive, and complete, in other words, it is assumed in Savage's system that all acts are pairwise comparable. SVG 2 can be easily derived from the completeness assumption and (P2), which says that the conditional preference relation over acts is definable for any given event and is complete. The next two postulates are commonly known as the "independence axioms" which impose further assumptions that the agent's probabilistic estimations over events and value judgments on consequences are, generally speaking, mutually independent: SVG 3 says that the preference ranking of constant acts is solely dependent on the values of their respective consequences which are robust against all states and SVG 4 says that the agent's qualitative probability estimations are independent of his value judgments over consequences (and that the relation "more probable" in Definition 5.4 is well defined). SVG 5 is brought in in order to rule out the trivial case where the agent is indifferent among all consequences (constant acts). With these postulates in hand, let's proceed to show the following preparatory results.

LEMMA 6.2. For any consequences $a, b \in X$ and for any event $E \in \mathcal{F}$, if $a \succsim b$ then $\mathbf{c}_a \succsim \mathbf{c}_a \oplus_E \mathbf{c}_b \succsim \mathbf{c}_b$.

PROOF. Given $a \succsim b$, we have that $\mathbf{c}_a \succsim \mathbf{c}_b$ by (5.3). Let E be any non-null event, then, by SVG 3, $\mathbf{c}_a \succsim_E \mathbf{c}_b$ (this holds trivially if E is a null event); from which we get, through (5.8), that for any $h \in \mathcal{A}$,

$$\mathbf{c}_a \oplus_E h \succsim \mathbf{c}_b \oplus_E h. \quad (6.1)$$

Let $h = \mathbf{c}_b$, that is $\mathbf{c}_a \oplus_E \mathbf{c}_b \succsim \mathbf{c}_b \oplus_E \mathbf{c}_b = \mathbf{c}_b$. This shows that $\mathbf{c}_a \oplus_E \mathbf{c}_b \succsim \mathbf{c}_b$. Similarly, one can show $\mathbf{c}_a \succsim \mathbf{c}_b \oplus_{E^C} \mathbf{c}_a$ by replacing h and E in (6.1) with \mathbf{c}_a and E^C , respectively. Then, by Lemma 5.2(1), $\mathbf{c}_a \succsim \mathbf{c}_a \oplus_E \mathbf{c}_b$. \square

LEMMA 6.3. For any $E, F \subseteq S$, if $F \subseteq E$ then $E \succeq F$.

PROOF. For any $a, b \in X$, assume that $a \succsim b$, and hence $\mathbf{c}_a \succsim \mathbf{c}_b$, then by Lemma 6.2, $\mathbf{c}_a \succsim \mathbf{c}_a \oplus_F \mathbf{c}_b$. By SVG 2, for \mathbf{c}_a and $\mathbf{c}_a \oplus_F \mathbf{c}_b$, at least one of the following two conditions holds,

- (i) $\mathbf{c}_a \oplus_F \mathbf{c}_b \succsim_E \mathbf{c}_a$;
- (ii) $\mathbf{c}_a \succsim_E \mathbf{c}_a \oplus_F \mathbf{c}_b$.

Suppose that (i) is the case, then, by (5.8), for any $h \in \mathcal{A}$, $(\mathbf{c}_a \oplus_F \mathbf{c}_b) \oplus_E h \succsim \mathbf{c}_a \oplus_E h$. Let $h = \mathbf{c}_a$, we have $\mathbf{c}_a \oplus_{E \cup F} \mathbf{c}_b \succsim \mathbf{c}_a$. On the other hand, for $E^C \cup F$, Lemma 6.2 implies that

$\mathfrak{c}_a \succsim \mathfrak{c}_b \oplus_{E^c \cup F} \mathfrak{c}_b$. Together, we have

$$\mathfrak{c}_a \oplus_{E^c \cup F} \mathfrak{c}_b \sim \mathfrak{c}_a. \quad (6.2)$$

Note that (6.2) holds for all $a, b \in X$ and all $E, F \subseteq S$ with $F \subseteq E$. Then let $E = S$ and $F = \emptyset$, from which we get $\mathfrak{c}_a \sim \mathfrak{c}_b$ for all $a, b \in \mathcal{A}$. But this is impossible if SVG 5 is in place.

The remaining possibility is (ii). In this case it follows, again by (5.8), that, for any $h \in \mathcal{A}$, $\mathfrak{c}_a \oplus_E h \succsim (\mathfrak{c}_a \oplus_F \mathfrak{c}_b) \oplus_E h$. Let $h = \mathfrak{c}_b$, we get $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim (\mathfrak{c}_a \oplus_F \mathfrak{c}_b) \oplus_E \mathfrak{c}_b$. Apply Lemma 5.2(2),

$$\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim (\mathfrak{c}_a \oplus_F \mathfrak{c}_b) \oplus_E \mathfrak{c}_b = \mathfrak{c}_a \oplus_{F \cap E} \mathfrak{c}_b = \mathfrak{c}_a \oplus_F \mathfrak{c}_b.$$

This yields that $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_F \mathfrak{c}_b$, hence, by Definition 5.4, $E \succeq F$. \square

THEOREM 6.4. If the preference relation \succsim on \mathcal{A} satisfies SVG 1-5 then the relation \succeq over events is a qualitative probability.

PROOF. We prove by direct verifications that \succeq as defined in Definition 5.4 satisfies conditions i-iv in Definition 6.1.

- i. Suppose that $E \succeq E' \succeq E''$, we show $E \succeq E''$. By definition, for any $a, b \in Z$ with $a \succsim b$, we have that $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E''} \mathfrak{c}_b$; then, by the transitivity of \succsim (SVG 1), we get $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E''} \mathfrak{c}_b$, this yields that $E \succeq E''$. Hence, \succeq is transitive. Completeness can be shown similarly.
- ii. In Lemma 6.3 let $F = \emptyset$, we get $E \succeq \emptyset$ for all $E \subseteq S$.
- iii. Let $a, b \in X$ be such that $\mathfrak{c}_a \succ \mathfrak{c}_b$ (i.e., $\mathfrak{c}_a \succsim \mathfrak{c}_b$ but $\mathfrak{c}_b \not\sucsim \mathfrak{c}_a$, the existence of the pair is guaranteed by SVG 5). Suppose, to the contrary, that $\emptyset \succeq S$. Then, by (5.4), $\mathfrak{c}_a \oplus_{\emptyset} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_S \mathfrak{c}_b$. On the other hand, note that $\mathfrak{c}_a \oplus_{\emptyset} \mathfrak{c}_b = \mathfrak{c}_b$ and $\mathfrak{c}_a \oplus_S \mathfrak{c}_b = \mathfrak{c}_a$, hence we have $\mathfrak{c}_b \succsim \mathfrak{c}_a$, a contradiction. Therefore, $S \succ \emptyset$.
- iv. Suppose $E \succeq E'$ and let F be such that $E \cap F = E' \cap F = \emptyset$, we show $E \cup F \succeq E' \cup F$. By definition, for any $a, b \in X$ with $a \succsim b$, we have that $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b$. Further, by SVG 2, one of the following is true,
 - (a) $\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b \succsim_{F^c} \mathfrak{c}_a \oplus_E \mathfrak{c}_b$;
 - (b) $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim_{F^c} \mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b$.

Suppose that (a) is the case, this implies that, for any $h \in \mathcal{A}$,

$$(\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b) \oplus_{F^c} h \succsim (\mathfrak{c}_a \oplus_E \mathfrak{c}_b) \oplus_{F^c} h$$

Since $E \cap F = E' \cap F = \emptyset$, let $h = \mathfrak{c}_b$, we get, via Lemma 5.2(2),

$$\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b = \mathfrak{c}_a \oplus_{E' \cap F^c} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E \cap F^c} \mathfrak{c}_b = \mathfrak{c}_a \oplus_E \mathfrak{c}_b$$

By definition, we have $E' \succeq E$. This, together with the assumption $E \succeq E'$, imply that for any $E, E' \subseteq S$, $E \succeq E'$ iff $E' \succeq E$, which contradicts (iii). The remaining

possibility is (b), for which we have that, for any $h \in \mathcal{A}$,

$$(\mathbf{c}_a \oplus_E \mathbf{c}_b) \oplus_{FC} h \succsim (\mathbf{c}_a \oplus_{E'} \mathbf{c}_b) \oplus_{FC} h.$$

Let $h = \mathbf{c}_a$. Then, by Lemma 5.2(1),

$$\mathbf{c}_a \oplus_F (\mathbf{c}_a \oplus_E \mathbf{c}_b) \succsim \mathbf{c}_a \oplus_F (\mathbf{c}_a \oplus_{E'} \mathbf{c}_b).$$

This yields, via Lemma 5.2(3), that $\mathbf{c}_a \oplus_{E \cup F} \mathbf{c}_b \succsim \mathbf{c}_a \oplus_{E' \cup F} \mathbf{c}_b$, and hence, by Definition 5.4, $E \cup F \succeq E' \cup F$. \square

This completes the proof that the “more probable” relation \succeq among events is indeed a qualitative probability. Before moving to show that there exists a unique probability measure that agrees with \succeq , let us explore some properties of qualitative probabilities which will become handy later.

COROLLARY 6.5. Let \succeq be as in Definition 5.4, then for any $E, E', F, F' \subseteq S$ the following hold:

- (1) if $F \succeq E$ and $F \cap F' = \emptyset$, then $F \cup F' \succeq E \cup F'$;
- (2) if $\emptyset \succeq E$, then $E \cup F' \simeq F'$;
- (3) if E is a null event, then $E \cup F \simeq F$;
- (4) if $F \succ E, F' \succ E'$ and $F \cap F' = \emptyset$, then $F \cup F' \succ E \cup E'$;
- (5) if $F \cup F' \succ E \cup E'$ and $E \cap E' = \emptyset$, then either $F \succ E$ or $F' \succ E'$;
- (6) if $E^C \succ E$ and $F \succ F^C$, then $F \succ E$.

PROOF. By Theorem 6.4, \succ is a qualitative probability, and hence satisfies conditions (a)–(d) in Definition 6.1.

- (1) Let $E_1 = E - F'$, then $E \cup F' = E_1 \cup F'$. From $E_1 \subseteq E$ and the assumption $F \succeq E$ it follows from Lemma 6.3 that $F \succeq E \succeq E_1$, hence $F \cup F' \succeq E_1 \cup F'$, that is, $F \cup F' \succeq E \cup F'$.
- (2) In (1) let $F = \emptyset$, then $F' \succeq E \cup F'$. On the other hand $E \cup F' \succeq F'$ via Lemma 6.3. Hence $E \cup F' \simeq F'$.
- (3) This is a direct consequence of (2) and Lemma 5.10(1).
- (4) Let $F = A \cup Q, E' = B \cup Q$ where $A = F - E', B = E' - F$ and $Q = F \cap E'$, hence $F \cup B = E' \cup A = A \cup B \cup Q$. Since $B \cap F = \emptyset$, it follows from the assumption $F \succeq E$ and (1) that $E' \cup A = F \cup B \succeq E \cup B$. On the other hand, $A \subseteq F$ and $F \cap F' = \emptyset$ hence $A \cap F' = \emptyset$, then from $F' \succ E'$ it follows that $F' \cup A \succeq E' \cup A$ via (1). Together we have $F' \cup A \succeq E \cup B$. Finally, since $Q \cap (F' \cup A) = \emptyset$ we have, again by (1),

$$F \cup F' = F' \cup A \cup Q \succeq E \cup B \cup Q = E \cup E'.$$

- (5) Otherwise, $E \succ F$ and $E' \succ F'$ which imply $E \succeq F$ and $E' \succeq F'$, then by (4), $E \cup E' \succeq F \cup F'$. It follows that $F \cup F' \simeq E \cup E'$ for all subsets E, E', F, F' of S with $E \cap E' = \emptyset$, which is absurd.
- (6) Let $\{A, B, C, D\}$ be a partition of S such that $F = A \cup B, F^C = C \cup D, E = A \cup C$, and $E^C = B \cup D$. By assumption $E^C = B \cup D \succeq C \cup A = E$, this implies, through (5), that either $B \succeq C$ or $D \succeq A$:
- i. If $B \succeq C$, it follows from the fact that A, B are disjoint that $B \cup A \succeq C \cup A$ via (1), and hence $F \succeq E$.
 - ii. If $D \succeq A$, also $F = A \cup B \succeq C \cup D = F^C$, then by (4) above, $A \cup B \cup D \succeq A \cup C \cup D$. It follows that $B \succeq C$ via (1), hence back to case (i).

Therefore, $F \succeq E$. □

REMARK 6.6. It is easy to verify that Corollary 6.5 (1) and (4)-(6) continue to hold with ' \succ ' in place of ' \succeq '.

The following observations are easy consequences of Corollary 6.5 which will be useful in the proof of the existence of numerical probability representation below.

COROLLARY 6.7. Let $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^n$ be partitions of S ,

- (1) if $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^n$ are so indexed that $E_1 \preceq \cdots \preceq E_n$ and $F_1 \succeq \cdots \succeq F_n$, then for any $r = 1, \dots, n$,

$$\bigcup_{j=1}^r F_j \succeq \bigcup_{i=1}^r E_i; \quad (6.3)$$

- (2) if in addition $E_i \simeq E_j$ and $F_i \simeq F_j$ for all $i, j \in \{1, \dots, n\}$, i.e., if $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^n$ partition S into n equally probable events, then

$$\bigcup_{i=1}^r E_i \simeq \bigcup_{j=1}^r F_j. \quad (6.4)$$

PROOF. (1) We prove by induction on r . Note that for the case $r = 1$, it must be that $F_1 \succeq E_1$. For, otherwise $E_1 \succ F_1 \succeq \cdots \succeq F_n$. It follows that $E_i \succ F_i$ for all $i = 1, \dots, n$, and hence $\bigcup_{i=1}^n E_i = S \succ \bigcup_{i=1}^n F_i = S$ (this is obtained by repeatedly applying the " \succ -version" of Corollary 6.5(4) since E_i 's are mutually disjoint), a contradiction.

In the inductive step, assume that (6.3) holds for r , we show that it holds for the case $r + 1$. Suppose, to the contrary, that $\bigcup_{i=1}^r E_i \cup E_{r+1} \succ \bigcup_{i=1}^r F_i \cup F_{r+1}$. Then, by the inductive hypothesis, $\bigcup_{i=1}^r F_i \succeq \bigcup_{i=1}^r E_i$, hence by Corollary 6.5(5) it must be that $E_{r+1} \succ F_{r+1}$. It follows that $E_i \succ F_i$ for all $i = r + 1, \dots, n$. This together with $\bigcup_{i=1}^{r+1} E_i \succ \bigcup_{i=1}^{r+1} F_i$ imply that $\bigcup_{i=1}^n E_i = S \succ \bigcup_{i=1}^n F_i = S$ via Corollary 6.5(4), which is impossible.

- (2) This is a direct consequence of (1) above. □

REMARK 6.8. Kraft et al. (1959) showed, through a counter example, that, contrary to what de Finetti (1951) had conjectured, the four conditions in Definition 6.1 are insufficient to bring about a numerical representation of \succeq in the sense of (5.5) even when $|S|$ is finite. In the same paper they gave the extra condition that is needed in order that the probable relation be represented by a numerical probability in finite cases (see also Scott, 1964). We shall not pursue this direction here. In what follows, we study Savage's approach to the problem, which is more general, for it also treats infinite cases.

6.2. Quantitative probability. In this section we show that the qualitative probability relation derived from SVG 1-5 in Theorem 6.4 admits a unique numerical representation provided that an additional postulate is inserted. That is, we show that there is a unique probability measure μ on (S, \mathcal{F}) such that

$$E \succeq F \iff \mu(E) \geq \mu(F), \quad \text{for all } E, F \in \mathcal{F}, \quad (6.5)$$

In this case, we say that the probability measure μ *agrees* with the qualitative probability \succeq , and say that μ *almost agrees* with \succeq if only the ' \Rightarrow ' direction of (6.5) holds. The representation rests on the following postulate.

SVG 6. For any $f, g \in \mathcal{A}$ and for any $a \in X$, if $f \succ g$ then there is a finite partition $\{P_i\}_{i=1}^n$ such that, for all i , $c_a \oplus_{P_i} f \succ g$ and $f \succ c_a \oplus_{P_i} g$.

The postulate says that if f is strictly preferred to g , then there exists a partition such that the preferential relation remains the same if f (g) is revised on the same cell of the partition with any constant act c_a . This postulate is a version of *continuity* axiom which is structurally similar to vNM 3 and A-A 3. It amounts to saying that the state space can be arbitrarily divided so that the revision of an act with respect to a constant act on any member of the partition is considered as preferentially insignificant. As we shall soon see, SVG 6 imposes sufficient structural constraint on the system that facilitates a numerical representation.

6.2.1. *Fineness and tightness.* As a show of the strength of SVG 6, let us first make the following observations.

LEMMA 6.9. Let \succeq be a qualitative probability satisfying SVG 6, and E, F be any events. Suppose that $F \succ E$, then there exists a partition $\{P_i\}_{i=1}^n$ ($n \leq \infty$) of S such that $F \succ E \cup P_i$, for all $i = 1, \dots, n$.

PROOF. By Definition 5.4, $F \succ E$ implies that, for any $a, b \in X$ with $a \succ b$, $c_a \oplus_F c_b \succ c_a \oplus_E c_b$. Now, in SVG 6 let $c_a \oplus_F c_b$ be in the place of f and let $c_a \oplus_E c_b$ be in that of g , then there exists a finite partition $\{P_i\}_{i=1}^n$ such that, for all i ,

$$c_a \oplus_F c_b \succ c_a \oplus_{P_i} (c_a \oplus_E c_b).$$

By Lemma 5.2(3), it follows that

$$c_a \oplus_F c_b \succ c_a \oplus_{E \cup P_i} c_b.$$

Hence, by definition, $F \succ E \cup P_i$. \square

LEMMA 6.10. Given any two events E and F , if, for any non-null events G, H satisfying $E \cap G = F \cap H = \emptyset$, $E \cup G \succeq F$ and $F \cup H \succeq E$, then $E \simeq F$.

PROOF. Suppose, to the contrary, that there exist E, F such that $E \succ F$ for all non-null G, H satisfying $E \cap G = F \cap H = \emptyset$, $E \cup G \succeq F$ and $F \cup H \succeq E$. Then by Lemma 6.9 there exists a partition $\{P_i\}_{i=1}^n$ ($n \leq \infty$) of S such that $E \succ F \cup P_i$ for all $i = 1, \dots, n$. For each P_i , if $F \cap P_i \neq \emptyset$ then split it into two cells $F - P_i$ and $P_i - F$, then we can refine partition $\{P_i\}_{i=1}^n$ with a new partition $\{P'_j\}_{j=1}^m$ such that, for each new cell P'_j one of the following conditions holds

$$F \cap P'_j = \emptyset \text{ or } P'_j \subseteq F. \quad (6.6)$$

Since each P'_j is a subset of some P_i , by Lemma 6.3, $E \succ F \cup P_i$ implies that

$$E \succ F \cup P'_j \text{ for all } j = 1, \dots, m.$$

Note that if $F \cap P'_j = \emptyset$ then P'_j must be null, for otherwise, by hypothesis, we have that $F \cup P'_j \succ E$, a contradiction. By (6.6), it follows that the only non-null cells of $\{P'_j\}_{j=1}^m$ are the ones contained in F , then, by Lemma 6.5(3), we have

$$E \succ F \succeq F \cup \bigcup_j P'_j = S$$

which is impossible. Hence $E \not\succeq F$. Similarly, it can be shown that $F \not\succeq E$. \square

Note that, in Lemma 6.9, let $E = \emptyset$, then we have that, for any $F \succ \emptyset$, there is a partition of S such that no element of which is as probable as F . In this case, we say that the qualitative probability \succeq is *fine*. The property presented in Lemma 6.10 is often referred to as the *tightness* condition of \succeq . The above shows that both fineness and tightness are guaranteed if SVG 6 is in place.

6.2.2. *Savage's triples.* Next we show some further consequences of SVG 6. These properties reveal some fine structures of the qualitative probability \succeq under SVG 1-6.

LEMMA 6.11. Let E, F, G, H, K be events, the following properties hold:

- (1) if $E \succ \emptyset$ then E can be partitioned into G, H where $G, H \succ \emptyset$;
- (2) if E, K, F are pairwise disjoint with $E \cup K \succ F \succeq E$, then K can be partitioned into G, H such that $E \cup H \succ F \cup G$;
- (3) if E, F are such that $E, F \succ \emptyset$ and $E \cap F = \emptyset$ then F can be partitioned into G, H for which $E \cup G \succeq H \succeq G$.

PROOF. (1) By SVG 6, there exists some partition $\{P_i\}$ such that $E \succ P_i$, and hence, by Lemma 6.3, $E \succ E \cap P_i$ for all $i = 1, \dots, n$. Suppose that $E \cap P_i \simeq \emptyset$ for all i 's, then by Corollary 6.5(2), $E \simeq \emptyset$, a contradiction. Suppose that there is only one P_i such that $E \cap P_i \not\simeq \emptyset$, then we have $E \simeq E \cap P_i$, again, a contradiction. Hence there are at least two cells P_i, P_j such that $E \cap P_i \not\simeq \emptyset$, $E \cap P_j \not\simeq \emptyset$, in which case let $G = E \cap P_i$ and $H = E - G$.

- (2) From the assumption $E \cup K \succ F \succeq E$ it is easy to see, via Corollary 6.5(2), that $K \succ \emptyset$. By SVG 6, there exists a n -partition $\{P_i\}$ such that $E \cup K \succ P_i \cup F$ for all i 's and that there must be one cell, say P_i , of the partition such that $K \cap P_i \succ \emptyset$, then we have $E \cup K \succ (K \cap P_i) \cup F$. Next, by (1), $K \cap P_i$ can be partitioned into G, G' with $G' \succeq G$, then we have

$$E \cup [(K - G) \cup G] = E \cup K \succ (K \cap P_i) \cup F = G' \cup G \cup F.$$

This yields $E \cup (K - G) \succ G' \cup F \succeq G \cup F$ (because $G' \succeq G$). Let $H = K - G$, then we get what we want.

- (3) If $E \succeq F$, by (1), F can be partitioned into $G, H \succ \emptyset$ with $H \succeq G$, in which case the claim follows trivially. Otherwise, $F \succ E \succ \emptyset$, then by SVG 6, there exists a n -partition $\{P_i\}$ such that $E \succ P_i$ and hence $E \succ P_i \cap F$ for $i = 1, \dots, n$. Rename $P_i \cap F$'s as Q_i 's and let latter be arranged such that $Q_1 \preceq Q_2 \preceq \dots \preceq Q_n$. Next, let m be such that

$$\bigcup_{i=1}^m Q_i \preceq \bigcup_{i=m+1}^n Q_i \preceq \bigcup_{i=1}^{m+1} Q_i \quad (6.7)$$

The existence of such an m is guaranteed by the assumption on $\{Q_i\}$ and the fact that \succeq is a qualitative probability. Then let $G = \bigcup_{i=1}^m Q_i$ and $H = \bigcup_{i=m+1}^n Q_i$. Then (6.7) yields $G \preceq H \preceq G \cup Q_{m+1}$. Since $E \cap F = \emptyset$ and $E \succ Q_{m+1}$ we get $E \cup G \succeq Q_{m+1} \cup G \succeq H$. \square

The existence of a numerical probability over events depends on the following construction, which is sometimes referred to as *Savage triples*.

LEMMA 6.12. There exists a sequence of 3-fold partitions $\{C_n, G_n, D_n\}_{n=1}^{\infty}$ of the state space S satisfying

- (1) $C_n \cup G_n \cup D_n = S$;
- (2) $C_n \cup G_n \succeq D_n$ and $D_n \cup G_n \succeq C_n$;
- (3) $C_n \subseteq C_{n+1}$, $D_n \subseteq D_{n+1}$, and $G_n \supseteq G_{n+1}$;
- (4) $G_n - G_{n+1} \succeq G_{n+1}$.

PROOF. By Lemma 6.11(1), S can be partitioned into $E, F \succ \emptyset$. Assume, WLOG, that $F \succeq E$ (otherwise, relabel the two events), then, by Lemma 6.11(3), F can be further partitioned into H, G such that $E \cup G \succeq H \succeq G$. Let $C_1 = E$, $G_1 = G$, and $D_1 = H$. Then

we have, for the case $n = 1$,

$$\begin{aligned} C_1 \cup G_1 \cup D_1 &= E \cup (G \cup H) = E \cup F = S, \\ C_1 \cup G_1 &= E \cup G \succeq H = D_1 \\ D_1 \cup G_1 &= (G \cup H) = F \succeq E = C_1. \end{aligned} \tag{6.8}$$

Next, consider the following cases

a. If $G_1 \simeq \emptyset$ we have, via Corollary 6.5(2), $C_1 \simeq D_1$ then it is plain that the claim is proved if we let $C_n = C_1$ and $D_n = D_1$ for all n 's.

b. If $G_1 \succ \emptyset$, we consider two subcases:

i. If $C_1 \cup G_1 \preceq D_1$, then we have, via (6.8), that $C_1 \cup G_1 \simeq D_1$. Apply Lemma 6.11(3) to C_1 and G_1 , we have that G_1 can be partitioned into H, G such that $C_1 \cup G \succeq H \succeq G$. In this case let $C_2 = C_1 \cup H$, $G_2 = G$, and $D_2 = D_1$, then we have

$$\begin{aligned} C_2 \cup G_2 \cup D_2 &= (C_1 \cup H) \cup G \cup D_1 = C_1 \cup G_1 \cup D_1 = S \\ C_2 \supseteq C_1, G_2 &\subseteq G_1, D_2 \supseteq D_1, \\ C_2 \cup G_2 &= (C_1 \cup H) \cup G = C_1 \cup G_1 \simeq D_1 = D_2, \\ D_2 \cup G_2 &= D_1 \cup G \simeq C_1 \cup G_1 \succeq C_1 \cup H = C_2, \\ G_1 - G_2 &= H \succeq G = G_2. \end{aligned} \tag{6.9}$$

ii. Now suppose $C_1 \cup G_1 \succ D_1$, also, from (6.8), we have $D_1 \cup G_1 \succeq C_1$. For the latter, if $D_1 \cup G_1 \simeq C_1$ then we are back to the previous case, otherwise we have $C_1 \cup G_1 \succ D_1$ and $D_1 \cup G_1 \succ C_1$. WLOG, assume that $C_1 \succeq D_1$, apply Lemma 6.11(2), we have that G_1 can be partitioned into H', G' such that $D_1 \cup H' \succeq C_1 \cup G'$. Further, by Lemma 6.11(3), H' can be partitioned into H, G such that $G' \cup H \succeq G \succeq H$. In this case, let $C_2 = C_1 \cup G'$, $G_2 = G$, and $D_2 = D_1 \cup H$, then we have

$$\begin{aligned} C_2 \cup G_2 \cup D_2 &= (C_1 \cup G') \cup G \cup (D_1 \cup H) = S \\ C_2 \supseteq C_1, G_2 &\subseteq G_1, D_2 \supseteq D_1, \\ C_2 \cup G_2 &= C_1 \cup G' \cup G \succeq D_1 \cup G \succeq D_1 \cup H = D_2, \\ D_2 \cup G_2 &= D_1 \cup H \cup G = D_1 \cup H' \succeq C_1 \cup G' = C_2, \\ G_1 - G_2 &= G' \cup H \succeq G = G_2. \end{aligned} \tag{6.10}$$

Repeat the above procedure for all $n \geq 2$, then we get what we want. \square

6.2.3. *Partition with equiprobable events.* One crucial step towards numerical probabilities is to show that, under SVG 1-6, the state space can be arbitrarily partitioned into equally probable events.

LEMMA 6.13. Let \succeq be a qualitative probability satisfying SVG 6, then S can be partitioned into 2^n ($n < \infty$) many equiprobable events.

PROOF. By Lemma 6.12, there exists a sequence of Savage-triples $\{C_n, G_n, D_n\}$. Then, for any event $E \succ \emptyset$, we have that $E \succeq G_n$ when n is large. For, otherwise, $G_n \succ E$ for all n . In this case let $\{P_i\}_{i=1}^m$ be an m -fold partition of S such that $E \succ P_i$ ($i = 1, \dots, m$) (the existence of such a partition is guaranteed by Lemma 6.9). We have $G_n \succ P_i$, for each i . Then, from conditions (3) and (4) above,

$$G_1 - G_2 \succeq \dots \succeq G_{n-2} - G_{n-1} \succeq G_{n-1} \succeq G_{n-1} - G_n \succeq G_n \succ P_i$$

By the ' \succ -version' of Corollary 6.5(4), it follows that $G_1 = (G_1 - G_2) \cup \dots \cup (G_{n-1} - G_n) \cup G_n \succ \bigcup_i P_i = S$, which is impossible. Hence $E \succeq G_n$. Then from this we conclude, via Lemma 6.3, that

$$E \succeq \bigcap_n G_n \quad \text{for any } E \succ \emptyset. \quad (6.11)$$

Now suppose that $\bigcap_n G_n \succ \emptyset$, then there exists a partition $\{P_i\}_{i=1}^m$ of S such that $\bigcap_n G_n \succ P_i$ for all i . Further let some P_j in the partition be such that $P_j \succ \emptyset$ (such an P_j must exist, otherwise we have $S = \bigcup_i P_i \simeq \emptyset$, which is impossible). But observe that if in (6.11) we let $E = P_j$, then it follows that $P_j \succeq \bigcap_n G_n \succ P_j$, a contradiction. Hence $\bigcap_n G_n \preceq \emptyset$.

Now take $S_1 = \bigcup_n C_n$ and $S_2 = \bigcup_n D_n \cup \bigcap_n G_n$. By Lemma 6.5(2) and the conclusion that $\bigcap_n G_n \preceq \emptyset$, we have that $S_1 \simeq S_2$ via condition (2) above. Hence $\{S_1, S_2\}$ equally partitions S . Apply the above procedure to S_1 and S_2 , and so on. Therefore, S can be partitioned into 2^n equivalent events for any n . \square

THEOREM 6.14. Let \succsim and \succeq be defined as above, then if \succsim satisfies SVG 1-6, there exists a unique (finitely additive) probability measure μ that represents \succeq :

$$E \succeq F \iff \mu(E) \geq \mu(F). \quad (6.12)$$

PROOF. We proceed in following steps.

(1) By Lemma 6.13, for any large $n \leq \infty$ in the form of 2^m for some m , there exists a partition $\{E_i\}_{i=1}^n$ of S such that $E_j \simeq E_k$ for all $j, k \in \{1, \dots, n\}$. Let $\mu(\cdot)$ be a real-valued function such that for each E_i

$$\mu(E_i) = \frac{1}{n} \quad i = 1, 2, \dots, n. \quad (6.13)$$

Now fix an event B , let r be the largest integer such that the union of r -many E_i 's is not more probable than B , that is,

$$\bigcup_{i=1}^{r+1} E_i \succ B \succeq \bigcup_{i=1}^r E_i. \quad (6.14)$$

Note that, for any fixed B , this integer r depends on n . However, as shown in Corollary 6.7(2), it is independent of the choice of n -fold partition of S . Let us denote r by a

function $k(B, n)$, we show that $\left\{ \frac{k(B, i)}{i} \right\}_{i=1}^{\infty}$ is a Cauchy sequence. To this end, suppose that F_1, \dots, F_m is an m -fold equal partition of S and t is the largest integer such that $\bigcup_{j=1}^{t+1} F_j \succ B \succeq \bigcup_{j=1}^t F_j$. Apply Lemma 6.13 again, we have that each $E_i (1 \leq i \leq n)$ and $F_j (1 \leq j \leq m)$ can be further partitioned, respectively, into m and n equally probable events, i.e., $E_i = \bigcup_{j=1}^m E_{ij}$ and $F_j = \bigcup_{i=1}^n F_{ji}$, where E_{ij} and F_{ji} are cells in the refined nm -fold equal partitions, then we have that

$$\bigcup_{j=1}^{t+1} \bigcup_{i=1}^n F_{ji} = \bigcup_{j=1}^{t+1} F_j \succ B \succeq \bigcup_{i=1}^r E_i \succeq \bigcup_{i=1}^{r-1} E_i \succeq \bigcup_{i=1}^{r-1} \bigcup_{j=1}^m E_{ij}.$$

Then, Corollary 6.7 implies that

$$(r-1)m \leq (t+1)n. \quad (6.15)$$

Here, by definition, $k(B, n) = r$ and $k(B, m) = t$, then (6.15) yields

$$\left| \frac{k(B, m)}{m} - \frac{k(B, n)}{n} \right| \leq \frac{1}{n} + \frac{1}{m} < \varepsilon,$$

where ε is an arbitrarily small number. The second inequality is met when m and n are sufficiently large. Hence, it is meaningful to define $\mu(B)$ by

$$\mu(B) =_{\text{Df}} \lim_{n \rightarrow \infty} \frac{k(B, n)}{n}. \quad (6.16)$$

(2) We need to verify that $\mu(\cdot)$ defined in (6.13) is a (finitely additive) probability measure, that is, μ satisfies the following conditions: for any E, F ,

- (a) $\mu(E) \geq 0$;
- (b) if $E \cap F = \emptyset$, then $\mu(E \cup F) = \mu(E) + \mu(F)$;
- (c) $\mu(S) = 1$.

Condition (a) and (c) can be easily verified. To show condition (b), let $\{P_i\}_{i=1}^n$ be an n -fold equal partitions of S , and let $r = k(E, n)$, $t = k(F, n)$, and $u = k(E \cup F, n)$. Since $E \cap F = \emptyset$, by Corollary 6.5(4) and (6.14),

$$\bigcup_{i=1}^{u+1} P_i \succ E \cup F \succeq \bigcup_{i=1}^r P_i \cup \bigcup_{j=1}^t P_j \quad (6.17)$$

Note that (6.17) hold even when the P_i 's and P_j 's on the right hand side are disjoint, and hence $\bigcup_{i=1}^{u+1} P_i \succeq \bigcup_{j=1}^{r+t} P_j$. It follows that $r+t \leq u+1$. On the other hand,

$$\bigcup_{i=1}^{r+t+2} P_i \succ E \cup F \succeq \bigcup_{i=1}^u P_i \quad (6.18)$$

This shows $u \leq r+t+2$ (To see the first inequality, note that otherwise we have $E \cup F \succeq \bigcup_{i=1}^{r+1} P_i \cup \bigcup_{i=1}^{t+1} P_i$, then by Corollary 6.5(5), either $E \succeq \bigcup_{i=1}^{r+1} P_i$ or $F \succeq \bigcup_{i=1}^{t+1} P_i$).

But neither case is possible). Hence

$$\frac{k(E, n)}{n} + \frac{k(F, n)}{n} - \frac{1}{n} \leq \frac{k(E \cup F, n)}{n} \leq \frac{k(E, n)}{n} + \frac{k(F, n)}{n} + \frac{2}{n}.$$

Let $n \rightarrow \infty$ we obtain that $\mu(E \cup F) = \mu(E) + \mu(F)$, which is what we want.

- (3) Finally, we show that μ defined in (6.16) is unique. Consider otherwise, then let μ' be another probability measure on S such that (6.5) holds. It follows, via (6.14), that $\frac{k(B, n)}{n} \leq \mu'(B) \leq \frac{k(B, n) + 1}{n}$. Now let $n \rightarrow \infty$, we get $\mu'(B) = \lim_{n \rightarrow \infty} \frac{k(B, n)}{n} = \mu(B)$. This shows uniqueness. \square

One feature of the probability measure μ derived in the theorem above is that μ is *atomless*. That is, as the following corollary shows, it allows for partitions of the state space into sets of arbitrarily small probability.¹¹

COROLLARY 6.15. Given the probability measure μ on S obtained above, for any $B \subseteq S$ and $0 \leq \rho \leq 1$, there exists $C \subseteq B$ such that $\mu(C) = \rho\mu(B)$.

PROOF. The proof is trivial if B is null. Now assume that $\mu(B) = p > 0$. By Lemma 6.13 and Theorem 6.14, for any large n in the form of 2^m , there exists a partition $\{E_i\}_{i=1}^n$ of B and unique probability measure μ on S for which

$$\mu(E_i) = \frac{p}{n} \quad \text{for all } i = 1, \dots, n$$

Now, let r be the largest number such that

$$\frac{(r+1)p}{n} > \rho \geq \frac{rp}{n}.$$

Define A_n, B_n by

$$A_n = \bigcup_{i=1}^r E_i, \quad B_n = \bigcup_{i=1}^{r+1} E_i.$$

Then we have

$$\mu(B_n) = \frac{p(r+1)}{n} > p\rho \geq \frac{pr}{n} = \mu(A_n).$$

By Theorem 6.14, $\mu(A_n) = \mu(B_n) = p\rho$ as $n \rightarrow \infty$. Define $C = \lim_{n \rightarrow \infty} A_n$. Then we have that $\mu(C) = \rho\mu(B)$. \square

REMARK 6.16. Intuitively, Corollary 6.15 says that, for any event B receiving non-zero probability under μ , B can be infinitely and continuously divided. As a consequence of this feature, the state space S in Savage's decision model must contain *uncountably many states*. This, however, sets a limit to application of Savage's theory: it cannot be applied to cases with a finite or countable state space.

¹¹See also Savage (1972, p.34) and Fishburn (1970, p.199).

7. Personal Utility

7.1. Utilities for simple acts. Next, we seek to construct a utility function for acts. This was approached by Savage in two steps. First, he considers a special set of simple acts, or *gambles* in his terminology, which are acts that potentially lead to only *finitely* many possible consequences, for which a von Neumann-Morgenstern utility function (vNMU) over consequences can be derived. The latter together with the derived subjective probability μ above give rise to a utility measure U_0 of simple acts. He then extends this utility for simple acts to general acts which can lead to potentially *infinitely* many consequences. The exposition here follows Savage's original approach, in Chapter ?? we will provide an alternative method of deriving utilities without appealing to constant-acts.

Let us start with a close examination of relationship between gambles and the class of lotteries as introduced in §2.3.

DEFINITION 7.1 (Gambles). An act $f \in X^S$ is said to be *simple* if there exist

- (i) a n -partition $\{P_i\}_{i=1}^n$ of S , and
- (ii) a finite sequence of consequences x_1, x_2, \dots, x_n such that $f(s) = x_i$ for all $s \in P_i$ ($i = 1, \dots, n$).

Denote the set of all simple acts by \mathcal{A}_0 , we also refer to simple acts as *gambles*. It is plain that all constant acts \mathbf{c}_x ($x \in X$) are gambles/simple acts. Using our notation for compound acts, a gamble $f \in \mathcal{A}_0$ can be conveniently expressed by

$$f = \mathbf{c}_{x_1} \oplus_{P_1} (\mathbf{c}_{x_2} \oplus_{P_2} (\mathbf{c}_{x_3} \oplus_{P_3} (\dots \oplus_{P_{n-1}} \mathbf{c}_{x_n}) \dots)). \quad (7.1)$$

7.1.1. Lotteries introduced by gambles. Now, given the subjective probability μ on S derived from Theorem 6.14, each gamble $f \in \mathcal{A}_0$ defines a simple probability measure on X , written p_f , as follows

$$p_f(x_i) = \begin{cases} \mu[f(s) = x_i] & \text{if } x_i \in f(S), \\ 0 & \text{if } x_i \in X - f(S); \end{cases} \quad (7.2)$$

where $\mu[f(s) = x_i] = \mu\{s \in S \mid f(s) = x_i\}$ and $f(S)$ denotes the range of f (cf. §2.1). We refer to p_f as *the lottery on X introduced by gamble f* .

Recall that \mathcal{L}_X^* is the set of simple probability measures defined on (an infinite) X (see Definition 2.8). Thus each $f \in \mathcal{A}_0$ corresponds to a simple probability measure in the extended lottery space \mathcal{L}_X^* . Observe that two different gambles may introduce the same lottery. Take, for instance, E, E^C be a partition of S for which $\mu(E) = \mu(E^C) = 1/2$ and let f, g be two acts defined in the table below.

	E	E^C
f	x_1	x_2
g	x_2	x_1

Then, we have an example where $f \neq g$, yet, by (7.2), $p_f = p_g$. That is, f and g induce the same lottery. We show in the following lemma that this is the case *only if* $f \sim g$. Intuitively, the lemma says that a pair of simple acts are considered equally preferable if the probabilities of getting each consequence under either one of the two acts are the same. As we shall soon see, this is a crucial step moving towards the full expected utility theory.

LEMMA 7.2. For any gambles $f, g \in \mathcal{A}_0$, if $p_f = p_g$ then $f \sim g$.

PROOF. We consider only the case where $f(S) = g(S)$. For if $f(S) \neq g(S)$, that is, if there is some $x_0 \in X$ such that, say, $x_0 \in f(S)$ but $x_0 \notin g(S)$, then, by the assumption that $p_f = p_g$ and (7.2), we have $\mu[f(s) = x_0] = 0$. In this case, we can construct an act f' which differs from f only on the null set $E_0 = \{s \mid f(s) = x_0\}$ (and hence $f' \sim f$ by Lemma 5.10(3)) such that $f'(S) = f(S) - \{x_0\}$. Repeat this process until we reach some f^* and g^* such that $f^* \sim f, g^* \sim g$ and $f^*(S) = g^*(S)$.

Now let $D = f(S) = g(S)$. The lemma is proved by induction on the size of D . Suppose that $|D| = 1$, then f, g are constant acts and $f = g$, and hence $f \sim g$. For the inductive step, assume that claim holds for $n - 1$, we show that it also holds when $|D| = n$. To this end, let x_1, x_2, \dots, x_n be an enumeration of the consequences in D , and let $\{P_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^n$ be partitions of S such that

$$f(s) = g(t) = x_i \text{ for all } s \in P_i \text{ and } t \in Q_i \text{ (} i = 1, \dots, n \text{)}. \quad (7.3)$$

We proceed with the following two possibilities:

- (1) If for some j , P_j and Q_j are null events. It follows that $\mu(P_j) = \mu(Q_j) = 0$, and hence $\mu[f(s) = x_j] = \mu[g(t) = x_j] = 0$. In this case, let r be such that P_r, Q_r are non-null. Then construct new gambles f' and g' as follows

$$f'(s) = \begin{cases} x_i & \text{if } s \in P_i \text{ and } i \notin \{j, r\} \\ x_r & \text{if } s \in P_j \cup P_r \end{cases}; \text{ and}$$

$$g'(s) = \begin{cases} x_i & \text{if } s \in Q_i \text{ and } i \notin \{j, r\} \\ x_r & \text{if } s \in Q_j \cup Q_r \end{cases}.$$

That is to say, f' agrees with f on all cells of the partition $\{P_i\}_{i=1}^n$ except for the null cell P_j , in which $f(s) = x_j$ but $f'(s) = x_r$, same for g and g' . By Lemma 5.10(2), we

TABLE 7.1

	P_1	P_2						
Q_1	A	D	f					
Q_2	C	B	<table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border: 1px solid black;">x_1</td><td style="border: 1px solid black;">x_2</td></tr> <tr><td style="border: 1px solid black;">x_1</td><td style="border: 1px solid black;">x_2</td></tr> </table>	x_1	x_2	x_1	x_2	
x_1	x_2							
x_1	x_2							

			g
	x_1	x_1	
	x_2	x_2	

have that,

$$f \succsim g \iff f' \succsim g'.$$

From the construction of f' and g' it is easily seen that they are gambles with $n - 1$ partitions and that $f'(S) = g'(S) = D - \{x_j\}$. Then by the inductive hypothesis $f' \sim g'$, and hence $f \sim g$.

(2) The remaining case is that P_i, Q_i are not null for all $i = 1, \dots, n$. We deal with this case in yet another two steps:

(a) As an illustration, consider the simple situation where $n = 2$. In this case we have that $X = \{x_1, x_2\}$ and that $\{P_1, P_2\}$ and $\{Q_1, Q_2\}$ are partitions of S for which

$$f(s) = g(t) = x_i \text{ for all } s \in P_i, t \in Q_i \text{ (} i = 1, 2) \quad (7.4)$$

$$\mu(P_1) = \mu(Q_1), \mu(P_2) = \mu(Q_2). \quad (7.5)$$

We want to show that $f \sim g$. To this end, let $A = P_1 \cap Q_1, B = P_2 \cap Q_2, C = P_1 \cap Q_2, D = P_2 \cap Q_1$, then (7.4) can be represented in Table 7.1. (for instance, $f(s) = x_1$ if $s \in A$ or $s \in C$). Next construct f' and g' which agree, respectively, with f and g on C and D and with each other on A and B . Then by the sure-thing principle (P2) $f \succsim g$ iff $f' \succsim g'$. It is hence sufficient to show that $f' \sim g'$.

	A	B	C	D	
f	x_1	x_2	x_1	x_2	
g	x_1	x_2	x_2	x_1	
f'	x_2	x_2	x_1	x_2	
g'	x_2	x_2	x_2	x_1	

f'	<table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border: 1px solid black;">x_2</td><td style="border: 1px solid black;">x_2</td></tr> <tr><td style="border: 1px solid black;">x_1</td><td style="border: 1px solid black;">x_2</td></tr> </table>	x_2	x_2	x_1	x_2
x_2	x_2				
x_1	x_2				

g'	<table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border: 1px solid black;">x_2</td><td style="border: 1px solid black;">x_1</td></tr> <tr><td style="border: 1px solid black;">x_2</td><td style="border: 1px solid black;">x_2</td></tr> </table>	x_2	x_1	x_2	x_2
x_2	x_1				
x_2	x_2				

Note that (7.5) implies $\mu(C) = \mu(D)$, then, by Theorem 6.14, it must be

$$C \simeq D \quad (7.6)$$

One the other hand, f' and g' can be written as

$$f' = \mathbf{c}_{x_1} \oplus_C \mathbf{c}_{x_2}, \quad (7.7)$$

$$g' = \mathbf{c}_{x_1} \oplus_D \mathbf{c}_{x_2}. \quad (7.8)$$

By Definition 5.4, (7.6)-(7.8) imply that $f' \sim g'$.

(b) In general, let $\{P_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^n$ be the partitions of S with respect to f and g satisfying (7.3). Let $B = P_n \cap Q_n$, $C = Q_n - P_n$ and $D = P_n - Q_n$. By assumption, $\mu(Q_n) = \mu(P_n)$. This implies that $\mu(C) = \mu(D)$. We consider only the nontrivial case where $\mu(C) = \mu(D) > 0$. Further, C can be partitioned such that $C_i = Q_n \cap P_i$ ($i = 1, \dots, n-1$). And it is clear $f(s) = x_1$ for all $s \in C_i$ ($i = 1, \dots, n-1$). Next, let $\mu(C_1)/\mu(C) = \rho_1$, then by Corollary 6.15, there exists some $D_1 \subseteq D$ for which $\mu(D_1)/\mu(D) = \rho_1$, and hence $\mu(C_1) = \mu(D_1)$. It is easy to see that, by repeatedly applying Corollary 6.15, D can be partitioned into D_1, \dots, D_{n-1} for which

$$\mu(C_i) = \mu(D_i), \quad i = 1, \dots, n-1. \quad (7.9)$$

TABLE 7.2

	P_1	P_2	\dots	P_n	f				
				D_1					D_1
				D_2					D_2
				\vdots					\vdots
Q_n	C_1	C_2	\dots	B		x_1	x_2	\dots	x_n

Now construct an act h_1 such that it agrees with f on all parts of S except for C_1 and D_1 for which

$$h_1(s) = \begin{cases} x_n & \text{if } s \in C_1, \\ x_1 & \text{if } s \in D_1, \\ g(s) & \text{otherwise.} \end{cases}$$

Since $\mu(C_1) = \mu(D_1)$, using a similarly argument given in part (a) above, we conclude that $h_1 \sim f$. Repeat this process inductively we have that

$$h_{i+1}(s) = \begin{cases} x_n & \text{if } s \in C_2, \\ x_{i+1} & \text{if } s \in D_2, \\ h_i(s) & \text{otherwise,} \end{cases} \quad (i < n-1).$$

From the construction h_i 's we have that $h_{n-1}(s) = x_n$ for all $s \in C_i$ ($i = 1, \dots, n-1$) and $h_{n-1} \sim f$.

h_1				x_1		$\Rightarrow \dots \Rightarrow$				x_1
				D_2						x_2
				\vdots						\vdots
Q_n	x_n	x_2	\dots	x_n			x_n	x_n	\dots	x_n

The proof is completed if we show that $h_{n-1} \sim g$. To this end, note that h_{n-1} agrees with g on Q_n , and hence $h_{n-1} \sim_{Q_n} g$. In $S - Q_n$, there are only $n - 1$ many elements, then by the construction of h_{n-1} and the inductive hypothesis we have that $h_{n-1} \sim_{S-Q_n} f \sim_{S-Q_n} g$. Together, we have $h_{n-1} \sim g$, which is what we want. \square

7.1.2. *Gambles introduced by lotteries.* Conversely, each lottery $p \in \mathcal{L}_X^*$ can be associated with a gamble. To see this, let x_1, x_2, \dots, x_n be an enumeration of the members of X that are in the support of p , as defined in (2.18), and let $\{P_i\}_{i=0}^n$ be a partition of S such that

$$\mu(P_i) = \begin{cases} 0 & i = 0, \\ p(x_i) & i = 1, \dots, n. \end{cases} \quad (7.10)$$

Note that the existence of such a partition is guaranteed by the fact that, by Lemma 6.13, S can be partitioned into arbitrarily fine equal-probable events and that μ is a well defined finitely additive probability measure on S for which Corollary 6.15 holds. Now, given p and the corresponding $\{P_i\}_{i=0}^n$, define f_p as follows

$$f_p(s) = \begin{cases} x & s \in P_0, \\ x_i & s \in P_i \quad (i = 1, \dots, n), \end{cases} \quad (7.11)$$

where x is an arbitrary consequence that is not in the support of p . We refer to f_p as a *gamble introduced by lottery p* . The following observation says that, given any lottery q , let f_q be a gamble introduced by q as defined above, then the introduced lottery by f_q is equal to q . The proof is immediate from (7.2) and (7.11), and hence omitted.

LEMMA 7.3. For any $q \in \mathcal{L}_X^*$, $p_{f_q} = q$.

It shall be emphasized that, for any simple act $g \in \mathcal{A}_0$, it is in general *not* the case that $f_{p_g} = g$. As the the following example illustrates, this is due to the fact that, in general, more than one gambles can be associated with the same lottery.

EXAMPLE 7.4. Let $X = \{x_1, x_2, x_3\}$ and p be such that $p(x_1) = p(x_2) = 0$ and $p(x_3) = 1$. Construct f and g to be such that $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ are their respective partitions of S for which $\mu(P_3) = \mu(Q_3) = 1$. By definition, both f and g are gambles introduced by p , but $f \neq g$.

$$\begin{array}{ccc} P_1 & P_2 & P_3 \\ \hline f & x_1 & x_1 & x_3 \end{array} \quad \begin{array}{ccc} Q_1 & Q_2 & Q_3 \\ \hline g & x_2 & x_2 & x_3 \end{array}$$

However, in the light of Lemma 7.2 and Lemma 7.3, we note that *all gambles introduced from the same lottery are equally preferable under \succsim* . It follows that each lottery $p \in \mathcal{L}_X^*$ can

be identified with a class of equally preferable gambles introduced by p , which are ordered under the given preference \succsim on \mathcal{A} .¹² For each p , let f_p be a representative of the associated equivalence class (under \succsim), then a preference relation \mathcal{L}_X^* can be *induced* as follows: for any $p, q \in \mathcal{L}_X^*$,

$$p \succsim q \text{ if } f_p \succsim f_q. \quad (7.12)$$

We show that this induced preference on \mathcal{L}_X^* satisfies von Neumann-Morgenstern axioms (cf. Remark 2.4).

LEMMA 7.5. If preference relation \succsim on \mathcal{A} satisfies SVG 1-6, then the induced ordering on \mathcal{L}_X^* in (7.12) satisfies the following conditions:

- (1) \succsim is a complete preference relation;
- (2) For all $p, q, r \in \mathcal{L}_X^*$ and $\lambda \in (0, 1]$, $p \succ q$ if and only if $p \oplus_\lambda r \succ q \oplus_\lambda r$;
- (3) For any $p, q, r \in \mathcal{L}_X^*$, if $p \succsim r \succsim q$ and $p \succ q$, then there exists a unique $\alpha \in [0, 1]$ such that $r \sim p \oplus_\alpha q$.

PROOF. (1) This is immediate from (7.12) and SVG 1.

- (2) By the definition of induced preference in (7.12), it is sufficient to show that the introduced gambles satisfy

$$f_p \succ f_q \text{ if and only if } f_{p \oplus_\lambda r} \succ f_{q \oplus_\lambda r}. \quad (7.13)$$

To this end, let $\{P_i\}_{i=0}^m, \{Q_j\}_{j=0}^r, \{R_k\}_{k=0}^n$ be partitions of S with respect to f_p, f_q, f_r , respectively, for which (7.10) and (7.11) are satisfied. By Corollary 6.15, construct $E_{ik} \subseteq P_i \cap R_k$ such that $\mu(E_{ik}) = \lambda \mu(P_i \cap R_k)$. Further, let $\bar{E}_{ik} = (P_i \cap R_k) - E_{ik}$, and hence $\mu(\bar{E}_{ik}) = (1 - \lambda)\mu(P_i \cap R_k)$. It follows that

$$\mu\left(\bigcup_k E_{ik}\right) = \lambda \mu(P_i) \quad \text{and} \quad \mu\left(\bigcup_i \bar{E}_{ik}\right) = (1 - \lambda)\mu(R_k). \quad (7.14)$$

It is plain that $\{E_{ik}, \bar{E}_{ik}\}_{ik}$ forms a finer partition of S . Define a gamble f_1 to be such that

$$f_1(s) = \begin{cases} x_i & \text{if } s \in E_{ik} \\ x_k & \text{if } s \in \bar{E}_{ik} \end{cases},$$

where x_i is in the support of p and $f_p(s) = x_i$ for $s \in E_{ik} \subseteq P_i$ and similarly, x_k is in the support of r and $f_r(s) = x_k$ for $s \in \bar{E}_{ik} \subseteq R_k$. Now let p_{f_1} be the lottery introduced

¹² Savage (1972, p.71) uses $\sum_i \rho_i f_i$ to denote the class of simple acts for which, to use his notations, there exist partitions B_i of S such that $P(B_i) = \rho_i$ and $f(s) = f_i$ for $s \in B_i$. He further remarks that if a simple act f is such that "the consequences f_i will befall the person in case B_i occurs, then the value of f is independent of how the partition B_i is chosen."

by f_1 , then, by (7.14), for any $x_i \in X$,

$$\begin{aligned} p_{f_1}(x_i) &= \mu[f_1(s) = x_i] = \mu\left[\bigcup_j E_{ij} \cup \bigcup_j \bar{E}_{ji}\right] \\ &= \lambda\mu(P_i) + (1 - \lambda)\mu(R_i) \\ &= \lambda p(x_i) + (1 - \lambda)r(x_i) \\ &= (p \oplus_\lambda r)(x_i). \end{aligned} \tag{7.15}$$

By Lemma 7.3, $p_{f_{p \oplus_\lambda r}} = p \oplus_\lambda r$, it follows that $p_{f_1} = p_{f_{p \oplus_\lambda r}}$, hence $f_1 \sim f_{p \oplus_\lambda r}$ via Lemma 7.2.

Similarly, construct $F_{jk} \subseteq Q_j \cap R_k$ such that $\mu(F_{jk}) = \lambda\mu(Q_j \cap R_k)$, and let $\bar{F}_{jk} = (Q_j \cap R_k) - F_{jk}$. Then $\{F_{jk}, \bar{F}_{jk}\}_{jk}$ partitions S . Define a gamble f_2 to be such that

$$f_2(s) = \begin{cases} x_j & \text{if } s \in F_{jk} \\ x_k & \text{if } s \in \bar{F}_{jk} \end{cases}.$$

We have that $p_{f_2} = q \oplus_\lambda r = p_{f_{q \oplus_\lambda r}}$, and hence $f_2 \sim f_{q \oplus_\lambda r}$. Thus, by SVG 1, (7.13) is proved if it can be shown that $f_p \succ f_q$ if and only if $f_1 \succ f_2$.

Observe that, by (7.15), for any $x_i \in X$, $f_1(s) = x_i$ implies $s \in P_i \cup R_i$ and, similarly, $f_2(s) = x_i$ only if $s \in Q_i \cup R_i$. Further, since f_p, f_q, f_r satisfy (7.11), construct two sequences of gambles h_1, \dots, h_n and h'_1, \dots, h'_n as follows

$$h_1 = f_p \oplus_{R_1^c} f_r \quad \text{and} \quad h'_1 = f_q \oplus_{R_1^c} f_r \tag{7.16}$$

$$h_{i+1} = h_i \oplus_{R_{i+1}^c} f_r \quad \text{and} \quad h'_{i+1} = h'_i \oplus_{R_{i+1}^c} f_r \tag{7.17}$$

From the constructions of f_1, f_2 , it is easy to see that $f_1 = h_n$ and $f_2 = h'_n$. Finally, by the sure-thing principle, (7.16) and (7.17) imply that

$$\begin{aligned} h_1 \succ h'_1 &\iff f_p \succ f_q \\ h_{i+1} \succ h'_{i+1} &\iff h_i \succ h'_i \end{aligned}$$

Therefore, $f_1 \succ f_2$ if and only if $f_p \succ f_q$, which is what we want.

(3) This claim can be similarly proved. \square

By Theorem 2.10, if the induced preference \succsim on \mathcal{L}_X^* satisfies vNM axioms, then there exists a vNMUF u for all the consequences in X , and hence an *expected utility function* U_0 for gambles such that, for each $f \in \mathcal{A}_0$,

$$U_0[f] = \sum_{x \in X} \mu[f(s) = x]u(x) = \int_S u[f(s)]d\mu(s). \tag{7.18}$$

Thus, Lemma 7.5 and (7.18) lead to the following theorem.

THEOREM 7.6. Let S be a set of states, X be a set of consequences, \succsim be a preference over the set of acts $\mathcal{A} = X^S$, and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the set of gambles, then, if \succsim satisfies SVG 1-6, there exists a utility function U_0 such that, for any $f, g \in \mathcal{A}_0$,

$$f \succsim g \iff U_0[f] \geq U_0[g],$$

where $U_0[f] = \int u[f] d\mu$.

The next order of business is to extend the utility function obtained in Theorem 7.6 for simple acts to that for general acts, namely, to relax the restriction that acts being considered have only finitely many possible consequences, which will be the subject of the next subsection. Before moving on, we show that, for any general act g , if g is bounded by two simple acts then there exists a simple act/gamble that is equally preferable to g .

COROLLARY 7.7. Let $f_1, f_2 \in \mathcal{A}_0$ satisfying $f_1 \succ f_2$, and $g \in \mathcal{A}$. If $f_1 \succsim g \succsim f_2$, then there exists a $g_0 \in \mathcal{A}_0$ such that $g_0 \sim g$.

PROOF. Our proof here parallels the proof of Lemma 2.3(4). In the notation of (7.2) and (7.11), let p_{f_1} and p_{f_2} be the lotteries induced by f_1, f_2 , and $f_{p_{f_1} \oplus \lambda p_{f_2}}$ is a gamble introduced by some mixer of p_{f_1} and p_{f_2} . Consider the following two sets

$$\begin{aligned} A &:= \{x \in [0, 1] \mid f_{p_{f_1} \oplus x p_{f_2}} \succsim g\}; \\ B &:= \{x \in [0, 1] \mid g \succsim f_{p_{f_1} \oplus x p_{f_2}}\}. \end{aligned} \tag{7.19}$$

Let $\alpha_* = \inf A$ and $\alpha^* = \sup B$. Note that, for any $a > \alpha_*$, there must exist some $a' \in A$ such that $a > a' \geq \alpha_*$. Then by Lemma 7.5(2) and Lemma 2.3(3), $f_{p_{f_1} \oplus a p_{f_2}} \succ f_{p_{f_1} \oplus a' p_{f_2}} \succsim g$. This means

$$a > \alpha_* \implies a \notin B. \tag{7.20}$$

The contrapositive of (7.20) says that, for any a , $a \in B$ implies that $\alpha_* \geq a$, in other words, α_* is an upper bound of B . and hence $\alpha_* \geq \alpha^*$. Similarly, one can show that, for any a ,

$$\alpha^* > a \implies a \notin A \tag{7.21}$$

which leads to $\alpha^* \geq \alpha_*$. Now define $\alpha = \alpha_* = \alpha^*$. It can be similarly shown, by applying SVG 6, that it cannot be that $\alpha \notin A \cap B$. Finally, define $g_0 = f_{p_{f_1} \oplus \alpha p_{f_2}}$, we have $g_0 \sim g$. \square

7.2. Postulate 7 and utility extension to general acts. To extend the utility for simple acts to acts in general, Savage brought in one final postulate.

SVG 7. For any event $E \in \mathcal{F}$, if $f \succsim_E c_{g(s)}$ for all $s \in E$ then $f \succsim_E g$.

The postulate says that, for any event E , if the conditional preference of f given E is no less preferable to any of the constant acts constructed from the possible consequences

of g under each state in E , then f is weakly preferred to g given E . As seen, this postulate uses constant acts in a systematic way, which, as we have discussed in Section ??, can be troublesome due to the issue of the applicability of the notion of constant acts. For the time being, let us focus on the following structural development of utility extension. Savage (1972, p.78) first demonstrated that SVG 7 is not derivable from the first six postulates. This was done by constructing a model which satisfies all of SVG 1-6 but fails SVG 7.

EXAMPLE 7.8. Let $S = \mathbb{N}^+$ and $X = [0, 1)$ be the set of consequences, and λ be the finitely but not countably additive measure on positive integers given in Example A.4.7. For any act f , let $U[f] = \int_S u(f) d\lambda$ where $u(x) = x$ is a utility function on X and $V[f] = \lim_{\epsilon \rightarrow 0} \lambda[f(s) \geq 1 - \epsilon]$, and let

$$\begin{aligned} W[f] &= U[f] + V[f] \\ &= \int_S u(f(s)) d\lambda(s) + \lim_{\epsilon \rightarrow 0} \lambda[f(s) \geq 1 - \epsilon] \end{aligned} \quad (7.22)$$

Define $f \succsim^* g$ to mean that $W[f] \geq W[g]$. It is not difficult to verify that the defined \succsim^* satisfies SVG 1-6.¹³ Note that for any act g with a finite range, i.e. a gamble, $V[g] = 0$, in this case $W[g] = U[g]$ is a utility function like the one given in Theorem 7.6. To see SVG 7 is violated, let f, g be such that

$$f(x) = \begin{cases} 1 - 1/x & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases} \quad \text{and} \quad g(x) = \max \{3/4, f(x)\}.$$

Then it is easy to calculate that

$$W[f] = \frac{1}{2} + \frac{1}{2} = 1, \quad \text{and} \quad W[g] = \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \right) + \frac{1}{2} = \frac{11}{8}.$$

Hence $f \prec^* g$ by the definition of \succsim^* in terms of $W[\cdot]$ above. On the other hand, for any $s \in S$, we have $g(x) < 1$. This means that, for the constant act $c_{g(s)}$, $W[c_{g(s)}] < 1$, and hence $f \succ^* c_{g(s)}$, from which we conclude that $f \succsim^* c_{g(s)}$ for all $s \in S$, but this contradicts SVG 7 (taking $E = S$). \triangleleft

Savage then showed that with SVG 7 the utility function U_0 for simple acts can be extended to a utility function U for general acts. To this end, we first prove the following lemmas.

LEMMA 7.9. For any event E , if, for every consequence $a \in X$, $f \succsim c_a$ and $g \succsim c_a$, then $f \sim g$.

PROOF. The lemma is proved by simple applications of SVG 7. \square

¹³See Example 2.1 (and Lemma 1 & 2) in Seidenfeld and Schervish (1983).

LEMMA 7.10. For any $f \in \mathcal{A}$, if there exists some $a \in X$ and $c < \infty$ such that $c_a \succsim f$ and $u(f(s)) \leq c$ for all $s \in S$, then there exists some gamble $g_0 \in \mathcal{A}_0$ for which

$$g_0 \succsim f \quad \text{and} \quad U_0[g_0] \leq c, \quad (7.23)$$

where u, U_0 are as in Theorem 7.6.

PROOF. Suppose that $u(a) \leq c$, then we can define $g_0 = c_a$. Otherwise, $u(a) > c$, in this case, fix any $t \in S$ and let $f(t) = b \in X$, we have, by the hypothesis, $u[f(t)] = u(b) \leq c$. Let p^* be a probability mixer of a and b such that

$$p^*(a)u(a) + (1 - p^*(b))u(b) = c.$$

Let E, E^C be a partition of S such that $\mu(E) = p^*(a)$ and $\mu(E^C) = p^*(b)$, and define a gamble g_0 to be such that

$$g_0(s) = \begin{cases} a & \text{if } s \in E \\ b & \text{if } s \in E^C \end{cases}$$

From the construction, we have $U_0[g_0] = c$. Further, for any $s \in S$, we have $U_0[c_{f(s)}] \leq c$, and hence, by Theorem 7.6, $g_0 \geq c_{f(s)}$. That is, $g_0 \geq c_{f(s)}$ for all $s \in S$ then, by SVG 7, $g_0 \succsim f$. \square

A small change of the proof above lead to the following corollary.

COROLLARY 7.11. For any $f \in \mathcal{A}$ and for any event E , if there exists some $a \in X$ and $c < \infty$ such that $c_a \succsim_E f$ and $u(f(s)) \leq c$ for all $s \in E$, then there exists some gamble $g_0 \in \mathcal{A}_0$ for which

$$g_0 \succsim_E f \quad \text{and} \quad U_0[g_0] \leq c. \quad (7.24)$$

LEMMA 7.12. Let $\{P_i\}_{i=1}^n$ be a partition of S and $c_1, \dots, c_n < \infty$. Then, for any act $f \in \mathcal{A}$, if there is a gamble $h_0 \in \mathcal{A}_0$ such that $f \succsim h_0$ and $u(f(s)) \leq c_i$ for all $s \in P_i$, then

$$U_0[h_0] \leq \sum_{i=1}^n c_i \mu(P_i). \quad (7.25)$$

PROOF. We consider the following two cases:

(1) If, for each P_i , there is some a_i such that $c_{a_i} \succsim_{P_i} f$. Then, by Corollary 7.11, there exists some g_i for which

$$g_i \succsim_{P_i} f \quad \text{and} \quad U_0[g_i] \leq c_i \quad \text{for all } i = 1, \dots, n.$$

Define g_0 to be such that $g_0(s) = g_i(s)$ if $s \in P_i$. Then we have $g_0 \succsim f$, hence $g_0 \succsim h_0$. Since both h_0 and g_0 are gambles, by Theorem 7.6, we have

$$U_0[h_0] \leq U_0[g_0] \leq \sum_{i=1}^n c_i \mu(P_i).$$

In this case, (7.25) holds.

(2) Otherwise, for some P_i ,

$$f \succ_{P_i} c_a \quad \text{for all } a \in X. \quad (7.26)$$

We show that, in this case, f can be modified to some f' so that

1. for each P_i , there is some $b_i \in X$ such that $c_{b_i} \succ_{P_i} f'$,
2. there exists some gamble h_0 such that $f \succ f' \succ h_0$, and
3. $u(f'(s)) \leq c_i$ for all $s \in P_i$.

If such a f' exists, this will take us back to case (1) for which (7.25) holds, then we are done. To this end, let $x^*, x_* \in X$ be such that $x^* \succ x_*$ and $u(x_*) < c_i$ (the existence of such a pair is guaranteed by SVG 5 and the fact utility is unique up to some linear transformation). Fix any $a \in X$, then, by SVG 6, $f \succ_{P_i} c_a$ implies that there is some non-null $A \subseteq P_i$ such that

$$\begin{aligned} c_{x^*} \oplus_A f &\succ_{P_i} c_a, \\ c_{x_*} \oplus_A f &\succ_{P_i} c_a. \end{aligned} \quad (7.27)$$

It is clear, by SVG 2, that

$$c_{x^*} \oplus_A f \succ_{P_i} c_{x_*} \oplus_A f. \quad (7.28)$$

Note that (7.26) implies, via SVG 7, that $f \succ_{P_i} c_{x^*} \oplus_A f$ and $f \succ_{P_i} c_{x_*} \oplus_A f$. Further, it cannot be the case that $c_{x_*} \oplus_A f \succ_{P_i} c_b$ for all $b \in X$, for, otherwise, by Lemma 7.9, $f \sim_{P_i} c_{x_*} \oplus_A f$, and hence $c_{x_*} \oplus_A f \succ_{P_i} c_{x^*} \oplus_A f$, which contradicts (7.28). This means that there is some $b_i \in X$ such that $c_{b_i} \succ_{P_i} f \sim_{P_i} c_{x_*} \oplus_A f$. Let $f' = c_{x_*} \oplus_A f$, then this is what we need. \square

THEOREM 7.13 (Savage). If \succsim satisfies SVG 1-7 then there exists a utility function u on X and a probability function μ on events such that for any $f, g \in \mathcal{A}$,

$$f \succsim g \iff \int u[f(s)] d\mu \geq \int u[g(s)] d\mu. \quad (7.29)$$

PROOF. We prove the theorem in following steps:

(1) Under the derived utility u on X and μ on \mathcal{F} from Theorem 7.6, we define the utility U of a general act f by¹⁴

$$U[f] = \int u[f] d\mu = \sup \sum_i \left[\inf_{s \in P_i} u[f(s)] \right] \mu(P_i), \quad (7.30)$$

where sup ranges over all possible finite partitions P_i of S into \mathcal{F} -sets. The goal is then to show that such a utility U exists under SVG 1-7.

(2) Given any general act $f \in \mathcal{A}$, we consider the following possibilities:

(a) $c_a \succsim f \succsim c_b$ for some $a, b \in X$;

¹⁴Cf. Section A.6.

- (b) $f \succ c_a$ for all $a \in X$;
(c) $c_a \succ f$ for all $a \in X$.

For case (a), partition S into $\{P_i\}_{i=1}^n$ and let P_i 's be so arranged that, for any $s \in P_i$ ($i = 1, \dots, n$),

$$c_* + \frac{i-1}{n}(c^* - c_*) \leq u[f(s)] \leq c_* + \frac{i}{n}(c^* - c_*). \quad (7.31)$$

where c_* and c^* are respectively the greatest lower and least upper bounds of u .¹⁵ Then from the definition of U in (7.30), it is easily seen that

$$\sum_{i=1}^n \left[c_* + \frac{i-1}{n}(c^* - c_*) \right] \mu(P_i) \leq U[f] \leq \sum_{i=1}^n \left[c_* + \frac{i}{n}(c^* - c_*) \right] \mu(P_i). \quad (7.32)$$

On the other hand, by Corollary 7.7, there exists some g_0 such that $g_0 \sim f$. Then from (7.32) we conclude via Lemma 7.11 (and an apparent symmetric argument) that

$$\sum_{i=1}^n \left[c_* + \frac{i-1}{n}(c^* - c_*) \right] \mu(P_i) \leq U_0[g_0] \leq \sum_{i=1}^n \left[c_* + \frac{i}{n}(c^* - c_*) \right] \mu(P_i). \quad (7.33)$$

Then (7.32) and (7.33) lead to

$$U[f] = U_0[g_0] \quad \text{as } n \rightarrow \infty. \quad (7.34)$$

If (b) is the case, then by Lemma 7.9, all acts that satisfy (b) are equally preferable. In this case, it is easy to show that $U[f] = c^*$. And similarly, for case (c), it can be shown that $U[f] = c_*$.

- (3) Finally, observe that (7.29) holds if we consider a combination of cases where f and g are in situations (a)-(c) above. \square

¹⁵Theorem 1 on page 79 of Savage (1972) was proved under the assumption that both f, g are bounded. In fact, Theorem 14.5 of Fishburn (1970, p.206) shows that the utility function u derived in Theorem 7.6 is bounded under SVG 1-7. See the footnote on page 80 in Savage (1972).

APPENDIX A

Some Mathematical Details

Gathered here are some of the definitions and results used or referred to in the main texts. They deliver some more details that complement discussions above. References of the sources are given from time to time, but all mistakes are mine.

A.1. Binary relations. Let X be a nonempty set, a *binary relation* R on X is a set of ordered pairs of elements of X . Following a notational convention, we sometimes write $(x, y) \in R$ in the form of xRy . The following is a list of properties of R : for any $x, y, z \in X$,

reflexivity: xRx

irreflexivity: $\neg xRx$

symmetry: $xRy \Rightarrow yRx$

asymmetry: $xRy \Rightarrow \neg yRx$

antisymmetry: $(xRy \wedge yRx) \Rightarrow x = y$

transitivity: $(xRy \wedge yRz) \Rightarrow xRz$

negatively transitivity: $(\neg xRz \wedge \neg zRy) \Rightarrow \neg xRy$ or $xRy \Rightarrow (xRz \vee zRy)$

completeness: xRy or yRx .

DEFINITION A.1.1. Let R be a binary relation on X , R is

- (1) a *preorder* if it is reflexive and transitive;
- (2) a *weak order* (or *total order*) if it is a complete preorder;
- (3) a *partial order* if it is an antisymmetric preorder;
- (4) a *linear order* if it is a complete partial order.

For any given preorder \succsim , by the *symmetric part* of \succsim , denoted by \sim , we mean $\sim = \{(x, y) \in \succsim \mid x \succsim y \text{ and } y \succsim x\}$, and by the *asymmetric part* (i.e., the *strict part*) of \succsim , denoted by \succ , we mean $\succ = \{(x, y) \in \succsim \mid x \succsim y \text{ and } y \not\succsim x\}$.

DEFINITION A.1.2. A *preordered set* is a structure (X, \succsim) where X is a nonempty set and \succsim is a preorder on X . A preordered set is said to be a *poset* (X, \succeq) if \succeq is a partial order on X ; it is a *loset* (X, \geq) if \geq is a linear order on X .

A binary relation E on X is said to be an *equivalence relation* if it is reflexive, symmetric and transitive. For any $x \in X$, the *equivalence class* of x with respect to E is the set

$$[x]_E = \{v \in X \mid xEv\}.$$

The collection of all equivalence classes of X with respect to E , denoted by X/E is the *quotient set* of X with respect to E , that is, $X/E = \{[x]_E \mid x \in X\}$. It is plain that, for any given preorder set (X, \succsim) , \succsim induces a partial order \succeq on the quotient set X/\sim of X such that

$$\begin{aligned} [x]_{\sim} \succeq [y]_{\sim} &\text{ if and only if } x \succ y \\ [x]_{\sim} = [y]_{\sim} &\text{ if and only if } x \sim y. \end{aligned}$$

DEFINITION A.1.3. Let (X, \succsim) be a preordered set, any x, y of X are said to be \succsim -*comparable* if either $x \succsim y$ or $y \succsim x$, and they are \succsim -*incomparable* if they are not \succsim -comparable, that is, if $x \not\succeq y$ and $y \not\succeq x$, denoted by $x \bowtie y$ (some writers also use ' $x \parallel y$ ' for incomparability).

A.2. Non-measurable sets. The following example is due to [Vitali \(1905\)](#). It shows that there exist sets of real numbers that are not Lebesgue measurable.

EXAMPLE A.2.1 (Vitali). Define an equivalence relation \sim on \mathbb{R} by:

$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$

By the Axiom of Choice, there exists a set V of representatives from each equivalent class. Now consider the set $\{V + r \mid r \in \mathbb{Q}\}$, it has following two properties:

- (1) For any distinct rational numbers r_1, r_2 ,

$$V + r_1 \cap V + r_2 = \emptyset.$$

(Otherwise, $V + r_1$ and $V + r_2$ share some point $h_1 + r_1 = h_2 + r_2$, then $h_1 \sim h_2$. Since h_1, h_2 are representatives it follows $h_1 = h_2$, and hence $r_1 = r_2$, a contradiction.)

- (2) For any $x \in \mathbb{R}$, $x \in V + r$ for some $r \in \mathbb{Q}$, that is,

$$\mathbb{R} = \bigcup \{V + r \mid r \in \mathbb{Q}\}. \tag{A.2.1}$$

(For, x must lie in some equivalence class with a representative, say, h . Then, by definition, $x - h = r'$ for some $r' \in \mathbb{Q}$, hence $x \in V + r'$.)

We show that it cannot be the case that $V \in \mathcal{B}$. Note that if $V \in \mathcal{B}$ then it must be that $\mu(V) > 0$. For, otherwise, $\mu(V) = 0$, then $\mu(V + r) = 0$ for all $r \in \mathbb{Q}$, since μ is

translation-invariant. But, by (A.2.1) and countable additivity,

$$\mu(\mathbb{R}) = \mu\left(\bigcup_{r \in \mathbb{Q}} \{V + r\}\right) = \sum_{r \in \mathbb{Q}} \mu(V + r) = 0$$

which is impossible. Further, if $\mu(V) > 0$ then there must be some $(a, b]$ for which $\mu(V \cap (a, b]) = c$ for some $c > 0$. Again, by translation-invariance,

$$\mu(V \cap (a, b]) = \mu(V \cap (a, b] + r) = c \quad \text{for all } r \in \mathbb{Q}. \quad (\text{A.2.2})$$

On the other hand, consider all the rationals in $[0, 1]$, we have

$$\bigcup_{r \in \mathbb{Q} \cap [0, 1]} (V \cap (a, b]) + r \subseteq (a, b + 1].$$

It follows that

$$\sum_{r \in \mathbb{Q} \cap [0, 1]} \mu(V \cap (a, b]) + r \leq \mu(a, b + 1] = b + 1 - a. \quad (\text{A.2.3})$$

However, by (A.2.2), the left hand of (A.2.3) is the sum of countable many c 's which add to $+\infty$, a contradiction. \triangleleft

A.3. Szpilrajn extension theorem. The following result is due to Szpilrajn (1930). It shows that every partial ordering can be extended to a linear ordering.

THEOREM A.3.1. Let \succ be a strict partial order on a set X . Then there exists a strict total order $>$ on X that extends \succ .

PROOF. Let \mathcal{P} be the set of all the strict partial orders on X that extend \succ . Then it is plain that \mathcal{P} is partially ordered under \subseteq . Let \mathcal{C} be any chain in the poset (\mathcal{P}, \subseteq) , then $\bigcup \mathcal{C}$ is an upper bound of \mathcal{C} . To see this, we show that $\bigcup \mathcal{C}$ is irreflexive and transitive, and hence $\bigcup \mathcal{C} \in \mathcal{P}$. Suppose, to the contrary, that there is an $x \in X$ such that $(x, x) \in \bigcup \mathcal{C}$, this implies that there exists some $C \in \mathcal{C}$ for which $(x, x) \in C$, which contradicts the assumption that C is a strict partial order. As for transitivity, suppose that $(x, y), (y, z) \in \bigcup \mathcal{C}$, then there exist $C_1, C_2 \in \mathcal{C}$ such that $(x, y) \in C_1$ and $(y, z) \in C_2$. Since \mathcal{C} is totally ordered under \subseteq , assume, without loss of generality, that $C_1 \subseteq C_2$, we get that $(x, z) \in C_2$, and hence $(x, z) \in \bigcup \mathcal{C}$.

By Zorn's lemma, \mathcal{P} contains a maximal element \bar{P} , that is, for any $P \in \mathcal{P}$, $\bar{P} \subseteq P$ implies that $P = \bar{P}$. We claim that \bar{P} must be a complete relation on X . For, otherwise, there exist some $x, y \in X$ such that neither $x\bar{P}y$ nor $y\bar{P}x$ hold. In this case, define $\bar{P}' = \bar{P} \cup A$ where $A = \{x\} \cup \{z \mid z\bar{P}x\} \times \{y\} \cup \{z \mid y\bar{P}z\}$. Then it is clear that \bar{P}' is a strict partial order on X that properly extends \bar{P} , which contradicts the maximality of \bar{P} . Thus, \bar{P} is irreflexive, transitive, and complete. Finally, denote \bar{P} by $>$, we have that $>$ is a strict total order that extends \succ . \square

A.4. Existence of uniform distribution over natural numbers. A uniformly distributed probabilistic measure on natural numbers \mathbb{N} is of particular interest because (1) it serves a good purpose of delineating the difference between finite additivity and countable additivity; (2) its use is often tied to the notion of randomness: it amounts to saying that choose a number “at random.” The latter is commonly understood in the following relative frequentist interpretation of uniformity of natural numbers.

A.4.1. *Density function.* Let A be any subset of \mathbb{N} . For each number $n < \infty$, denote the number of elements in A that are less or equal to n by $A(n)$, that is,

$$A(n) = |A \cap \{1, \dots, n\}|. \quad (\text{A.4.1})$$

Define the *density* of A by the limit (if exists)

$$d(A) = \lim_{n \rightarrow \infty} \frac{A(n)}{n}. \quad (\text{A.4.2})$$

Let \mathcal{C}_d be the collection of all sets of natural numbers that have densities. The following properties of the density function are easy to verify.

- PROPOSITION A.4.1. (1) $d(\emptyset) = 0$ and $d(\mathbb{N}) = 1$.
 (2) For each natural number n , $d(\{n\}) = 0$.
 (3) For any finite $A \in \mathcal{C}_d$, $d(A) = 0$.
 (4) If $A, B, A \cup B \in \mathcal{C}_d$ and $A \cap B = \emptyset$, then $d(A \cup B) = d(A) + d(B)$.
 (5) If $A \in \mathcal{C}_d$, then, for any number n , $A + n \in \mathcal{C}_d$ and $d(A) = d(A + n)$, where $A + n = \{x + n \mid x \in A\}$.
 (6) The set of even numbers has density $1/2$, or more generally, the set of numbers that are divisible by $m < \infty$ has density $1/m$.

Notice that d is not defined for *all* subsets of \mathbb{N} (\mathcal{C}_d is not a field of natural numbers). We hence seek to extend d to a finitely additive probability measure μ so that μ is defined for all subsets of the natural numbers and that μ agrees with d on \mathcal{C}_d (Theorem A.4.6 below). One version of the extension theorem has been given by Rao and Rao (1983, Theorem 3.2.10).¹ The set-theoretic approach explicated in the next subsection is adapted from Hrbacek and Jech (1999, Ch. 11). We include this construction for completion, readers may proceed directly to Example A.4.7 below without losing much on the flow of the main argument.

A.4.2. *Filter and ultrafilter.* A *filter* on a nonempty set S is a collection \mathcal{F} of subsets of S such that

- (1) $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,
 (2) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$,

¹Kadane and O’Hagan (1995, Theorem 1) show that the monotonicity condition given by Rao and Rao (1983) in their extension theorem is also necessary, see also Schirokauer and Kadane (2007).

(3) if $X, Y \subseteq S$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.

EXAMPLE A.4.2. (1) A trivial filter $\mathcal{F} = \{S\}$.

(2) Let $A \subseteq S$, a *principal filter* generate by A is the collection $\{X \subseteq S \mid A \subseteq X\}$. In the case of natural numbers where $S = \mathbb{N}$, a principal filter generated by $n_0 < \infty$ is the collection \mathcal{F}_{n_0} of sets of numbers such that $X \in \mathcal{F}_{n_0}$ if and only if $n_0 \in X$.

(3) As for an example of a *nonprinciple filter*, let S an infinite set, the *Fréchet filter* on S is the collection

$$\mathcal{F} = \{X \subseteq S \mid S - X \text{ is finite}\}. \quad (\text{A.4.3})$$

That is, \mathcal{F} is the filter of all *cofinite* subsets of S . ◁

A filter \mathcal{U} is said to be an *ultrafilter* if, for each $X \subseteq S$, either $X \in \mathcal{U}$ or $S - X \in \mathcal{U}$. The following extension theorem (due to Tarski, 1930) is crucial to our construction of an finitely additive probability measure on $\mathcal{P}(\mathbb{N})$. The proof uses Zorn's lemma and is widely available (see, for instance, Jech, 2003, §7).

THEOREM A.4.3 (Tarski). Every filter can be extended to an ultrafilter.

Recall that our main concern in the last subsection is that the density function $d(\cdot)$ is not defined for all the subset of natural numbers, in other words, there exists some $A \subseteq \mathbb{N}$ such that the sequence $\{A(n)/n\}_{n=1}^{\infty}$ does not converge. The goal is to extend d to some measure so that (A.4.2) holds for all A 's. To this end, we define a general notion of *convergence in an ultrafilter*, which has the property that, given an ultrafilter of natural numbers, *every* bounded sequences converges. As we shall see, this leads to the extension of d to $\mathcal{P}(\mathbb{N})$ as required.

DEFINITION A.4.4. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and let \mathcal{U} be an ultrafilter on \mathbb{N} . For some $a \in \mathbb{R}$, $\{a_n\}_{n=1}^{\infty}$ is said to be *convergent in \mathcal{U} to a* (or a is a *\mathcal{U} -limit* of the sequence), written $a = \lim_{\mathcal{U}} a_n$, if for every small $\epsilon > 0$,

$$\{n \mid |a_n - a| < \epsilon\} \in \mathcal{U}. \quad (\text{A.4.4})$$

LEMMA A.4.5. Let \mathcal{U} be an ultrafilter on \mathbb{N} , then, for any bounded real sequence $\{a_n\}$, there exists a unique \mathcal{U} -limit.

PROOF. Since $\{a_n\}$ is bounded, for every $x < \infty$, let

$$A_x = \{n \mid a_n < x\}.$$

Further, let

$$a = \sup\{x \mid A_x \notin \mathcal{U}\}.$$

We show that $\lim_{\mathcal{U}} a_n = a$, that is, we show that, for any $\epsilon > 0$, (A.4.4) holds. Note that, for any $x < y$, $A_x \subseteq A_y$, hence if $A_x \in \mathcal{U}$ then $A_y \in \mathcal{U}$. Since a is the least upper bound

of x for which $A_x \notin \mathcal{U}$, we have $A_{a+\epsilon} \in \mathcal{U}$ but $A_{a-\epsilon/2} \notin \mathcal{U}$. Given that \mathcal{U} is an ultrafilter, the latter implies that $S - A_{a-\epsilon/2} \in \mathcal{U}$, that is,

$$S - A_{a-2\epsilon} = \left\{ n \mid a - \frac{\epsilon}{2} \leq a_n \right\} \in \mathcal{U}.$$

Since $A_{a+\epsilon} = \{n \mid a_n < a + \epsilon\} \in \mathcal{U}$ and $\{n \mid a - \epsilon/2 \leq a_n\} \subseteq \{n \mid a - \epsilon < a_n\}$, we have that $\{n \mid |a_n - a| < \epsilon\} = \{n \mid a_n < a + \epsilon\} \cap \{n \mid a - \epsilon < a_n\} \in \mathcal{U}$, and hence (A.4.4). To show uniqueness, note that if there is some $b \neq a$ such that $b = \lim_{\mathcal{U}} a_n$. Let $\epsilon = |a - b|$, then, by (A.4.4), both $A = \{n \mid |a_n - a| < \epsilon/2\}$ and $B = \{n \mid |a_n - b| < \epsilon/2\}$ are in \mathcal{U} . Clearly, $A \cap B = \emptyset$, and hence $B \subseteq S - A$. But this implies, from $B \in \mathcal{U}$ and the fact that \mathcal{U} is an ultrafilter, that $S - A$ is also in \mathcal{U} , which is impossible. \square

THEOREM A.4.6. There exists a finitely additive probability measure on all subsets of \mathbb{N} that extends the density function d .

PROOF. Let \mathcal{U} be a Fréchet ultrafilter on \mathbb{N} (the existence of \mathcal{U} is guaranteed by Example A.4.2 (3) and Theorem A.4.3). Define a measure μ on $\mathcal{P}(\mathbb{N})$ to be such that

$$\mu(A) = \lim_{\mathcal{U}} \frac{A(n)}{n}, \quad (\text{A.4.5})$$

where $A(n)$ is defined as in (A.4.1). By Lemma A.4.5, μ is well defined for all $A \in \mathcal{P}(\mathbb{N})$. Note that, for any A , if $d(A)$ exists, say $d(A) = a$, then $a = \mu(A)$. For, by definition, if for any small ϵ there exists some N such that, for all $n > N$, $|A(n)/n - a| < \epsilon$, then, given that \mathcal{U} is the ultrafilter of all cofinite subsets of \mathbb{N} , it follows that $\{n \mid |A(n)/n - a| < \epsilon\} \in \mathcal{U}$, and hence $\mu(A) = a$.

It remains to show that μ is indeed a finitely additive probability measure. Clearly, $\mu(\emptyset) = 0$ and $\mu(\mathbb{N}) = 1$. We show μ is finitely additive. To this end, let A, B be any disjoint subsets of \mathbb{N} . By (A.4.5) and the fact that $A \cap B = \emptyset$,

$$\begin{aligned} \mu(A \cup B) &= \lim_{\mathcal{U}} \frac{(A \cup B)(n)}{n} \\ &= \lim_{\mathcal{U}} \frac{A(n) + B(n)}{n} \\ &= \lim_{\mathcal{U}} \frac{A(n)}{n} + \lim_{\mathcal{U}} \frac{B(n)}{n} = \mu(A) + \mu(B). \end{aligned}$$

(Actually, it can also be easily seen that μ is also translation-invariant.) Therefore, μ is a measure defined for all subsets of \mathbb{N} that extends the density function d . \square

The following is a classical example of finitely but not countably additive probability measure on the natural numbers which is a simple form of the density function d introduced above.

EXAMPLE A.4.7. Let $\{\lambda_n\}$ be a sequence of functions defined on \mathbb{N} such that²

$$\lambda_n(i) = \begin{cases} 1/n & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n. \end{cases} \quad (\text{A.4.6})$$

Clearly, each $\lambda_n(i)$ takes the form of $A(n)/n$ in (A.4.2) where $A = \{i\}$, and $\{\lambda_n\}$ converges point-wisely to the density function d (on singletons). By Theorem A.4.6, there exists a function λ defined for all subsets of \mathbb{N} that extends d . Further, by Proposition A.4.1, λ satisfies the following properties:

- (1) λ is defined for all subsets of \mathbb{N} .
- (2) $\lambda(\emptyset) = 0$ and $\lambda(\mathbb{N}) = 1$.
- (3) λ is finitely additive.
- (4) λ is *not* countably additive.
- (5) For any $i < \infty$, $\lambda(\{i\}) = 0$.
- (6) For any $A \subseteq \mathbb{N}$, if A is finite then $\lambda(A) = 0$; if A is cofinite (i.e. if $\mathbb{N} - A$ is finite) then $\lambda(A) = 1$.
- (7) $\lambda(\{2n \mid n \in \mathbb{N}\}) = 1/2$, i.e., the set of even numbers has measure $1/2$.
- (8) In general, the set of numbers that are divisible by $m < \infty$ has measure $1/m$, that is, $\lambda(\{1m, 2m, 3m, \dots\}) = 1/m$. As a result of this property, we have that the assignment of μ can be arbitrarily small: for any $\lambda > 0$, there exists some n such that the set of numbers that are divisible by n has measure $1/n < \epsilon$. \triangleleft

A.5. Convergences. Let $\{f_n\}, f$ be measurable functions on the measure space $(\Omega, \mathcal{F}, \mu)$,

- (1) f_n is said to *converges point-wisely* to f , in symbols $f_n \rightarrow f$, if

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega), \quad \text{for all } \omega \in \Omega. \quad (\text{A.5.1})$$

- (2) f_n is said to *converges uniformly* to f if, for any $\epsilon > 0$, there is some large N such that

$$\left| f_n(\omega) - f(\omega) \right| < \epsilon, \quad \text{for all } \omega \in \Omega, n \geq N. \quad (\text{A.5.2})$$

- (3) f_n is said to converge to f *almost everywhere (a.e.)* if there exists a measurable set $E \subseteq \Omega$ satisfying

$$\mu(E) = 0 \quad \text{and} \quad \lim_n f_n(\omega) = f(\omega) \quad \text{for all } \omega \in \Omega - E. \quad (\text{A.5.3})$$

- (4) f_n is said to converge to f *in measure* if

$$\lim_{n \rightarrow \infty} \mu \left[|f_n - f| \geq \epsilon \right] = 0 \quad \text{for all } \epsilon > 0. \quad (\text{A.5.4})$$

²Dubins and Savage (1965) call probability measure of this type *diffuse*.

LEMMA A.5.1. Given any finitely additive measure μ on measurable space (Ω, \mathcal{F}) , if f_n converges to 0 almost everywhere implies that f_n converges to 0 in measure, then the measure is also countably additive.

PROOF. Assume that $\{B_i\}$ is any sequence of pairwise disjoint sets in the measurable space, define

$$B = \bigcup_i B_i = \bigcup_{i \leq n} B_i \cup \bigcup_{i > n} B_i$$

Let $A_n = \bigcup_{i > n} B_i$, hence $A_n \downarrow \emptyset$. We show that $\mu(A_n) \rightarrow 0$ as $n \rightarrow \infty$. To this end, let χ_{A_n} be the characteristic function of A_n , it is plain that $\mu(A_n) \rightarrow 0$ if and only if $\chi_{A_n} \rightarrow 0$ in measure. By assumption, it is enough to ask $\chi_{A_n} \rightarrow 0$ *a.e.* but this follows trivially from the fact that $\bigcap A_n = \emptyset$. Next, note that, by finite additivity,

$$\mu(A_n) = \mu(B - \bigcup_{i \leq n} B_i) = \mu(B) - \sum_{i=1}^n \mu(B_i)$$

Hence, from $\mu(A_n) \rightarrow 0$, we get $\mu(B) = \sum_{i=1}^{\infty} \mu(B_i)$. This shows countable additivity. \square

A.6. Expectations. Let $\{f_n\}, \{g_n\}$ f, g be real-valued measurable functions on the measure space $(\Omega, \mathcal{F}, \mu)$. f is said to be *simple* if there n -many distinct values c_1, \dots, c_n and a partition $\{P_i\}_{i=1}^n$ of Ω such that $f(x) = c_i$ for all $x \in P_i$ ($i = 1, \dots, n$). Define the expectation of f with respect to μ to be

$$E(f, \mu) = \sum_{i=1}^n c_i \mu(P_i). \quad (\text{A.6.1})$$

DEFINITION A.6.1 (Expectation). If f is bounded and $\{f_n\}$ is a sequence of simple measurable functions converges uniformly to f then

$$E(f, \mu) = \sup \{E(f_n, \mu) : n = 1, 2, \dots\}. \quad (\text{A.6.2})$$

It can be shown that the above definition does not depend on the selection of the sequences of simple functions converging to f . As shown below, any bounded measurable function f can be approximated by a particular sequence of *simple* functions, and hence in (A.6.2) we can use this sequence of simple functions to calculate the expectation of f through (A.6.1). Suppose that $c_* \leq f \leq c^*$, for each $n < \infty$, define a n -partition $\{P_i\}_{i=1}^n$ of Ω to be such that

$$P_i = \left\{ x \mid c_* + \frac{(i-1)(c^* - c_*)}{n} \leq f(x) \leq c_* + \frac{i(c^* - c_*)}{n} \right\}, \quad (\text{A.6.3})$$

and define f_n by

$$f_n(x) = c_* + \frac{(i-1)(c^* - c_*)}{n} \quad \text{for all } x \in P_i. \quad (\text{A.6.4})$$

For each n , f_n is a simple function by definition, then we have that for all $x \in \Omega$,

$$|f(x) - f_n(x)| \leq \frac{c_* - c^*}{n}. \quad (\text{A.6.5})$$

Hence $\{f_n\}$ uniformly converges to f , in which case we have

$$E(f, \mu) = \sup \left\{ \sum_{i=1}^n \left[\inf_{x \in P_i} f(x) \right] \mu(P_i) : n = 1, 2, \dots \right\}. \quad (\text{A.6.6})$$

Note that the requirement of uniform convergence is crucial for those measure spaces with mere finitely additive probabilities. The following is the example commonly used in the literature to illustrate this point.

EXAMPLE A.6.2. Let $\Omega = \{0, 1, 2, \dots\}$ and λ be a diffuse (Example A.4.7) defined on (Ω, \mathcal{F}) and $f(x) = x/(1+x)$ for all $x \in \Omega$. Using the construction from (A.6.3) to (A.6.6), we can define a sequences $\{f_n\}$ of functions uniformly converging to f such that

$$f_n(x) = \frac{i-1}{n} \quad \text{for all } x \in P_i = \left\{ x \mid \frac{i-1}{n} \leq f(x) \leq \frac{i}{n} \right\} (i = 1, \dots, n).$$

Since, for each $i < n$, P_i is finite and hence $\lambda(P_i) = 0$, then we have

$$\begin{aligned} E(f, \lambda) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i-1}{n} \lambda(P_i) \\ &= \lim_{n \rightarrow \infty} \left[\sum_{i=1}^{n-1} \frac{i-1}{n} \lambda(P_i) + \frac{n-1}{n} \lambda(P_n) \right] \\ &= \lim_{n \rightarrow \infty} \frac{n-1}{n} = 1. \end{aligned}$$

Now consider another sequence g_n of functions constructed as follows. Let Q_i be a $n-1$ -partition of Ω such that

$$\begin{aligned} Q_1 &= \left\{ x \mid 0 \leq f(x) \leq \frac{1}{n} \right\} \cup \left\{ x \mid \frac{n-1}{n} \leq f(x) \leq 1 \right\} \\ Q_i &= \left\{ x \mid \frac{i-1}{n} \leq f(x) \leq \frac{i}{n} \right\} \quad (i = 2, \dots, n-1). \end{aligned}$$

Define g_n by

$$g_n(x) = \inf \left\{ f(y) \mid y \in Q_i \right\} \quad \text{for all } x \in Q_i (i = 1, \dots, n-1).$$

We have that, for each n , g_n is a simple function and $E(g_n, \lambda) \equiv 0$, and $\sup\{E(g_n, \lambda) : n = 1, 2, \dots\} = 0 \neq E(f, \lambda) = 1$. Note that the difference between $\{f_n\}$ and $\{g_n\}$ is that the latter does not converge uniformly to f . \triangleleft

A.7. Gambler's Ruin and Countable Additivity. The story we are about to tell points to one important source of the *countable additivity* condition for probability measures. The issue is closely related to the modern philosophical debate about finite versus countable additivity. As we shall see, countable additivity is needed even at very early stage of the development of probability theory.

The Gambler's Ruin. The original Gambler's Ruin is the problem posed by Pascal to Fermat through a letter from Carcavi to Huygens on 28 September 1656 (cf. [Hald, 2003](#), p.76). The problem goes like this: A and B are playing a game which involves the rolling of three fair dice. Each player is given 12 counters as his initial capital. The rule of the game is that if 11 points are shown, A gives a counter to B and if 14 points are shown, B gives a counter to A , then whoever first collects all the counters wins the game. The question is which one of the two players is more prone to win the game.³

Solution. Let us modernize the story: suppose that a gambler enters a game with capital a and adopts the strategy of continuing to bet at unit stakes with chance p of winning each bet (and chance $q = 1 - p$ of losing a bet) until his fortune increases to c or his funds are exhausted. Then the question is what is the probability of his achieving goal?

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables taking on $+1$ or -1 as values with probabilities $\Pr[X_n = +1] = p$ and $\Pr[X_n = -1] = q$. Define

$$\begin{aligned} S_0 &= 0; \\ S_n &= X_1 + \dots + X_n. \end{aligned}$$

Intuitively, S_n counts the wins of the gambler in the first n bets, and his fortune after the n 's bet is $a + S_n$. The event that the gambler achieved his goal after n 's round of betting can be described as

$$A_{a,n} = [a + S_n = c] \cap \bigcap_{k=1}^{n-1} [0 < a + S_k < c], \quad (\text{A.7.1})$$

where $[0 < a + S_n < c]$ represent the set of sequences of rolling such that the gambler's goal is not reached in the first k tries. It is easy to see that $m \neq n$ implies $A_{a,n} \cap A_{a,m} = \emptyset$. Then the probability of the gambler winning the game with capital a and goal c , denoted by $s_c(a)$, is

$$\begin{aligned} s_c(a) &= \sum_{n=1}^{\infty} \Pr(A_{a,n}) = \Pr\left(\bigcup_{n=1}^{\infty} A_{a,n}\right); \\ s_c(0) &= 0, \quad s_c(a) = 1. \end{aligned} \quad (\text{A.7.2})$$

³The mathematical detail below is pulled mainly from [Billingsley \(2012, §2 and §7\)](#).

Now apply Huyens' idea of shifting the betting sequence one step to the right, that is, from X_1, X_2, \dots to X_2, X_3, \dots . Then, the initial game is equivalent to a game where the gambler has either probability p to start betting with a capital of $a + 1$ or probability q with capital $a - 1$. This generates the following recursive function,

$$s_c(a) = ps_c(a + 1) + qs_c(a - 1). \quad (\text{A.7.3})$$

Assuming $0 \leq a \leq c$, let $r = q/p$, then the above equation can be solved as⁴

$$s_c(a) = \begin{cases} \frac{r^a - 1}{r^c - 1} & \text{if } r \neq 1 \\ a/c & \text{if } r = 1 \end{cases}. \quad (\text{A.7.4})$$

NOTE. The construction above involves

- (1) infinite sequences of observable results: in (A.7.1), $A_{a,n}$ (as $n \rightarrow \infty$) is the set of infinite sequence of rollings;
- (2) countably additivity probability: in (A.7.2), the probability of winning the game with initial capital a and goal c is the *sum* of probabilities of winning the game after n rollings ($n = 1, 2, \dots$).

Hence the solution in (A.7.4) is justifiable only if it can be shown the the underlying probability measure is (1) definable for infinite sequences and (2) is countable additive, to which we now turn.

Sequence space. Let S be a (finite) set of possible outcomes and ρ be a (simple) probability function defined on S . In the example above, $S = \{1, 2, 3, 4, 5, 6\}$ and $\rho(i) = 1/6$, for all $i \in S$. Let $\Omega = S^\infty$ and, for any $\omega \in \Omega$, let $z_k(\omega) : S^\infty \rightarrow S$ be the k th coordinate projection function for all $k \geq 1$. Define a cylinder of rank n to be a set of the form

$$A = [\omega : (z_1(\omega), \dots, z_n(\omega)) \in H],$$

where $H \subseteq S^n$. Let \mathcal{C}_0 be the class of cylinders of all ranks, then it is easy to verify that \mathcal{C}_0 is a field. The goal is to define a probability measure on measurable space (Ω, \mathcal{C}_0) . Now consider the following set function $\text{Pr}(\cdot)$ on \mathcal{C}_0 defined by

$$\text{Pr}(A) = \sum_H \rho(z_1(\omega)) \times \dots \times \rho(z_n(\omega)). \quad (\text{A.7.5})$$

We show that Pr is a well defined probability measure on (Ω, \mathcal{C}_0) . For this, we only show that $\text{Pr}(\cdot)$ is *finitely additive*: let A be as above and

$$B = [\omega : (z_1(\omega), \dots, z_m(\omega)) \in I]$$

⁴See also DeGroot and Schervish (2012, p.87).

for some $I \subseteq S^m$. WLOG, assume that $n \leq m$, then let $H' \subseteq S^m$ be such that, for each $\omega \in \Omega$, $(z_1(\omega), \dots, z_n(\omega), \dots, z_m(\omega)) \in H'$ iff $(z_1(\omega), \dots, z_n(\omega)) \in H$, and hence

$$A = [\omega : (z_1(\omega), \dots, z_m(\omega)) \in H']$$

Now suppose that $A \cap B = \emptyset$, then by (A.7.5)

$$\begin{aligned} \Pr(A \cup B) &= \sum_{H' \cup I} \rho(z_1(\omega)) \cdots \rho(z_m(\omega)) \\ &= \sum_{H'} \rho(z_1(\omega)) \cdots \rho(z_m(\omega)) + \sum_I \rho(z_1(\omega)) \cdots \rho(z_m(\omega)) \\ &= \Pr(A) + \Pr(B). \end{aligned}$$

$\Pr(\cdot)$ is referred to as a (finitely additive) *product measure* on (Ω, \mathcal{C}_0) . We point out that \Pr is at the same time countably additive. To this end, we first turn to the following observations

LEMMA A.7.1. If $\Pr(\cdot)$ is a finitely additive probability measure on the field \mathcal{F} , and if $A_n \downarrow \emptyset$ for sets A_n in \mathcal{F} implies $\Pr(A_n) \downarrow 0$, then $\Pr(\cdot)$ is countably additive.

PROOF. Assume that $\{B_n\}$ is a sequence of pairwise disjoint sets in \mathcal{F} , define

$$B = \bigcup_i B_i = \bigcup_{i \leq n} B_i \cup \bigcup_{i > n} B_i$$

Let $A_n = \bigcup_{i > n} B_i$, then $A_n \downarrow \emptyset$ as $n \rightarrow \infty$. Note that, by finite additivity,

$$\Pr(A_n) = \Pr(B - \bigcup_{i \leq n} B_i) = \Pr(B) - \sum_{i=1}^n \Pr(B_i)$$

Hence, from $\Pr(A_n) \rightarrow 0$, we get $\Pr(B) = \sum_{i=1}^{\infty} \Pr(B_i)$. □

LEMMA A.7.2. If $A_n \downarrow A$, where A_n are nonempty cylinders, then $A \neq \emptyset$.

PROOF. See Billingsley (2012, p.30) □

THEOREM A.7.3. Every finitely additive product measure on \mathcal{C}_0 is countably additive.

PROOF. Assume, to the contrary, that $\Pr(\cdot)$ is not countably additive, then apply Lemma A.7.1: there is some sequence $\{A_n\}$ in \mathcal{C}_0 such that $A_n \downarrow \emptyset$ and $\Pr(A_n)$ does not converge to 0, that is, there is some $\epsilon > 0$ for which $\Pr(A_n) > \epsilon$ as $n \rightarrow \infty$. This implies, by Lemma A.7.2, $\emptyset = A = \bigcap_n A_n \neq \emptyset$, which is absurd. □

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