NOTES ON FIXED-POINT THEOREMS

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In combinatorial topology, an *n-simplex* is an n-dimensional polytope which is the convex hull of its $n+1$ vertices. Formally, let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^{n+1}$ be the vertices, then the n-simplex, denoted by Δ^n, of $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ would be

$$\Delta^n = \Big\{ \sum_{i=0}^{n} \alpha_i \mathbf{v}_i \;\Big|\; \sum_{i=0}^{n} \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for all } i \Big\}.$$

Given any n-simplex Δ^n, the convex hull of any $m(\leq n)$ internal points of Δ^n that define a simplex is called a *face* of Δ^n. A *simplicial subdivision* of an n-simplex is a partition of Δ^n into *sub-simplices* (also called "cells") such that any two sub-simplices are either disjoint or they share a face of certain dimension. A *Sperner labeling* of a simplicial subdivision of an n-simplex is an assignment of numbers $0, 1, \ldots, n$ to the vertices of the subdivision, so that the vertices of Δ^n receive all different numbers (a *full Sperner labeling*), and points on each face of Δ^n use only the numbers of the vertices defining the respective face of Δ^n. A sub-simplex of Δ^n is said to be *distinguished* if it has a full Sperner labeling.

Lemma 1 (Sperner)**.** The number of distinguished sub-simplices of any simplex is odd.

Proof. Let Δ^n be any simplex, we prove by induction on the dimensionality of Δ^n.

1**-dimension:** Assume that $n = 1$, then we have the case of line segments. WLOG, let $\Delta^1 = [a, b]$, $a, b \in \mathbb{R}$. Divide $[a, b]$ into subintervals (sub-simplices). Applying the Sperner labeling, the endpoints a, b of Δ^1 are labelled, respectively, 0 and 1; the interval points of subdivision are marked either 0 or 1. Hence, a distinguished sub-simplex is just a subinterval with one of its endpoints labelled 0 and the other 1. Then, given the simplicial subdivision, let A be set of all distinguished sub-simplices and B the set of all non-distinguished sub-simplices, it is plain that $A \cup B$ contains all the sub-simplices and $A \cap B = \emptyset$.

Next, for each subinterval α, let $\nu(\alpha)$ be the number of its endpoints (faces) with label 0. It is easy to see that

$$\nu(\alpha) = \begin{cases} 1, & \text{if } \alpha \in A; \\ 0 \text{ or } 2, & \text{if } \alpha \in B. \end{cases}$$

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From which we get that $|A| = \sum_{\alpha \in A} \nu(\alpha)$ and that $\sum_{\alpha \in B} \nu(\alpha)$ is an even number. One the other hand, note that each internal endpoint is a face of two joint sub-intervals, say, α and β, hence if this endpoint is marked 0 then it will be counted twice in computing $\nu(\alpha)$ and $\nu(\beta)$, the only endpoint with label 0 which is counted once is just a. Thus if we add all the counts of 0's for all the sub-intervals we have that $\sum_{\alpha \in A \cup B} \nu(\alpha)$ must be an odd number. Note that

$$\begin{aligned}\sum_{\alpha \in A \cup B} \nu(\alpha) &= \sum_{\alpha \in A} \nu(\alpha) + \sum_{\alpha \in B} \nu(\alpha) \\ &= |A| + \sum_{\alpha \in B} \nu(\alpha) \qquad (1)\end{aligned}$$

where $\sum_{\alpha \in A \cup B} \nu(\alpha)$ is a odd number and $\sum_{\alpha \in B} \nu(\alpha)$ is an even number, from which we conclude that $|A|$ must be odd.

n**-dimension:** In the inductive step, assume the claim holds for Δ^{n-1}, that is, the number of distinguished sub-simplices of any $n-1$-simplex is odd. Then, for an n-simplex Δ^n, apply an simplicial subdivision of Δ^n with Sperner labeling. As before, let A be the set of all distinguished sub-simplices of the subdivision and B be the set of all non-distinguished ones. For each sub-simplex α, let $\nu(\alpha)$ denote the number of faces whose vertices are marked with *all* of $0, 1, \ldots, n-1$, and we call a face of this kind an n-face. Note that for any $\alpha \in B$, $\nu(\alpha) = 0$ if no face of α is an n-face, otherwise we have $\nu(\alpha) = 2$. This is because from $\alpha \in B$ we get that the labeling of the vertices of α uses at most n different numbers (otherwise $\alpha \in A$) so if α has a n-face this means that there is a number that has been used twice in labeling the vertices of α (there are $n+1$ many vertices of α). Hence, again we have

$$\nu(\alpha) = \begin{cases} 1, & \text{if } \alpha \in A; \\ 0 \text{ or } 2, & \text{if } \alpha \in B. \end{cases}$$

Similarly, from which we get that $|A| = \sum_{\alpha \in A} \nu(\alpha)$ and that $\sum_{\alpha \in B} \nu(\alpha)$ is an even number. Thus, to show $|A|$ is odd, by (1), it suffices to show that $\sum_{\alpha \in A \cup B} \nu(\alpha)$ is an odd number. To see this, obverse that for any n-face there are two possibilities: (1) the face is contained inside Δ^n, in this case, this face must be shared with another sub-complex, say, β, and hence it will counted twice in computing $\nu(\alpha)$ and $\nu(\beta)$; (2) the face is on the boundary of Δ^n, in this case, this n-face is just a distinguished sub-simplex of Δ^{n-1}. Let X be the number of n-faces in case (1) and Y be the number of n-faces in (2). Then it is plain that $\sum_{\alpha \in A \cup B} \nu(\alpha) = 2X + Y$. But by the inductive hypothesis, Y is odd, from which we conclude that $\sum_{\alpha \in A \cup B} \nu(\alpha)$ is an odd number, this completes the proof of the lemma. □

Theorem 2 (Brouwer)**.** Every continuous function f from a convex compact subset K of $\mathbb{R}^n$ to K itself has a fixed point.

Proof. We prove the simple case where K is an n-simplex embedded in $\mathbb{R}^{n+1}$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^{n+1}$ for which

$$\begin{aligned}\mathbf{v}_0 &= (1,0,0,\ldots,0)\\ \mathbf{v}_1 &= (0,1,0,\ldots,0)\\ &\vdots\\ \mathbf{v}_n &= (0,0,0,\ldots,1)\end{aligned}$$

By convexity, for any $\mathbf{x} \in K$, there is a sequence $\theta_0, \theta_1, \ldots, \theta_n \in \mathbb{R}$ for which $\sum_{i=0}^n \theta_i = 1$ such that $\mathbf{x} = \sum_{i=0}^n \theta_i \mathbf{v}_i = (\theta_0, \theta_1, \ldots, \theta_n)$, we use $(\mathbf{x})_i$ to express the ith coordinate of $\mathbf{x}$, i.e., $(\mathbf{x})_{i+1} = \theta_i$. We prove that, given the continuous function f, there exists an $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$.

Suppose, to the contrary, that there is no such fixed point, we show, by applying the Sperner's lemma above, that this is impossible. To this end, let us first construct a simplicial subdivision $\mathcal{S}_1$ of K. The Sperner labeling in the present context is defined as follows:

(1) Each vertex $\mathbf{v}_i$ of K is marked number i, $0 \le i \le n$.
(2) Let X be any sub-simplex of K in $\mathcal{S}_1$ and $\mathbf{v}^{(1)}$ be a vertex of X, then, there exist a sequence $\{\theta_j\}$ and a sequence $\{\delta_j\}$ for which $\sum_{i=0}^n \theta_i = \sum_{i=0}^n \delta_i = 1$ such that $\mathbf{v}^{(1)} = (\theta_0, \theta_1, \ldots, \theta_n)$ and $f(\mathbf{v}^{(1)}) = (\delta_0, \delta_1, \ldots, \delta_n)$. Label $\mathbf{v}^{(1)}$ with the smallest j such that $\delta_j < \theta_j$.
(Note that such an j must exist, for, otherwise, we have $\delta_i \ge \theta_i$ for all i, yet from the assumption that there is no fixed point we get that it cannot be the case that $\delta_i = \theta_i$ for all i, so there must be some k's for which $\delta_k > \theta_k$, from which we get $1 = \sum_{i=0}^n \delta_i > \sum_{i=0}^n \theta_i = 1$, a contradiction.)

By the Sperner's lemma, there exists a sub-simplex K_1 of K in $\mathcal{S}_1$ which has a full Sperner labeling. Since the labels of the vertices of K_1 uses all the different numbers of $\{0, 1, \ldots, n\}$ we can list all the vertices of K_1 using their Sperner labels as the subscripts of the vertices: $\mathbf{v}_0^{(1)}, \mathbf{v}_1^{(1)}, \ldots, \mathbf{v}_n^{(1)}$ (i.e., $\mathbf{v}_i^{(1)}$ is labeled number i in subdivision $\mathcal{S}_1$). Then by the instruction of Sperner labeling give above we have

$$\left(f(\mathbf{v}_i^{(1)})\right)_{i+1} < \left(\mathbf{v}_i^{(1)}\right)_{i+1} \quad \text{for all } i \in \{0, 1, \ldots, n\}.$$

Repeat the above procedure m many times: form a simplicial subdivision $\mathcal{S}_2$ of all the sub-simplices of K with the same Sperner labeling instruction, then, by the Sperner's Lemma, there exists a distinguished sub-simplex K_2 contained in K_1 with vertices $\mathbf{v}_0^{(2)}, \mathbf{v}_1^{(2)}, \ldots, \mathbf{v}_n^{(2)}$ such that $\left(f(\mathbf{v}_i^{(2)})\right)_{i+1} < \left(\mathbf{v}_i^{(2)}\right)_{i+1}$ for all $i \in \{0, 1, \ldots, n\}$, and so on.

By compactness, it is easily seen that the size of the distinguished sub-simplex K_m goes to 0 as $m \to \infty$, and the vertices $\mathbf{v}_0^{(m)}, \mathbf{v}_1^{(m)}, \ldots, \mathbf{v}_n^{(m)}$ will all converge to the same point, say, $\mathbf{x}^*$, as $m \to \infty$. Further, since f is continuous, we have that $f\left(\mathbf{v}_i^{(m)}\right) \to f(\mathbf{x}^*)$ as $m \to \infty$ for all $i \in \{0, 1, \ldots, n\}$, and from the construction above we have also that

$$\left(f\left(\mathbf{x}^*\right)\right)_j \leq \left(\mathbf{x}^*\right)_j \quad \text{for all } j \;\; 1 \leq j \leq n+1. \tag{2}$$

The assumption that there is no fixed point implies that there must be some k's such that

$$\left(f\left(\mathbf{x}^*\right)\right)_k < \left(\mathbf{x}^*\right)_k \tag{3}$$

From (2) and (3) we get $1 = \sum_{j=0}^n \left(f\left(\mathbf{x}^*\right)\right)_j < \sum_{j=0}^n \left(\mathbf{x}^*\right)_j = 1$, a contradiction. □

Definition 3. Let X be a compact convex subset of $\mathbb{R}^n$, a set function $f : X \to 2^{\mathbb{R}^n}$ is said to be *upper semi-continuous* if, for any $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$, $\mathbf{x}_n \to \mathbf{x}, \mathbf{y}_n \in f(\mathbf{x}_n)$ and $\mathbf{y}_n \to \mathbf{y}$ imply $\mathbf{y} \in f(\mathbf{x})$. Or, equivalently, f is upper semi-continous iff f has a closed graph (i.e., iff $\{(\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x})\}$ is closed in the product topology).

Theorem 4 (Kakutani)**.** Let K be a compact convex subset of $\mathbb{R}^n$ and $f : K \to 2^{\mathbb{R}^n}$ for which the set $f(\mathbf{x})$ is nonempty and convex for all $\mathbf{x} \in K$; and f is upper semi-continuous. Then, there exists $\mathbf{x}^* \in K$ such that $\mathbf{x}^* \in f(\mathbf{x}^*)$.

Proof. Assume that K is a simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$. Form the kth simplicial subdivision $\mathcal{S}_k$ and define $f^{(k)}$ as follows:

$$f^{(k)}(\mathbf{x}) = \begin{cases} \mathbf{y} \in f(\mathbf{x}) & \text{if } \mathbf{x} \in \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}, \\ \sum_{i=0}^n \theta_i \mathbf{y}_i & \text{if } \mathbf{x} = \sum_{i=0}^n \theta_i \mathbf{v}_i \text{ with } \sum_{i=0}^n \theta_i = 1, \text{ where } \mathbf{y}_i \in f(\mathbf{v}_i). \end{cases}$$

By convexity, $f^{(k)}(\mathbf{x}) \in f(\mathbf{x})$ and it is plain that, for each k, $f^{(k)} : K \to K$ is continuous, and by the Brouwer fixed-point theorem, there exists a fixed-point, call it $\mathbf{x}^{(k)}$, such that $\mathbf{x}^{(k)} = f^{(k)}(\mathbf{x}^{(k)})$. Note that $\mathbf{x}^{(k)}$ lies in some cell of $\mathcal{S}_k$ with vertices $\mathbf{v}_0^{(k)}, \mathbf{v}_1^{(k)}, \ldots, \mathbf{v}_n^{(k)}$, and hence there is a sequence $\theta_0^{(k)}, \theta_1^{(k)}, \ldots, \theta_n^{(k)}$ for which

$$\mathbf{x}^{(k)} = \sum_{i=0}^n \theta_i^{(k)} \mathbf{v}_i^{(k)}.$$

Then from $\mathbf{x}^{(k)} = f^{(k)}(\mathbf{x}^{(k)})$ we get that

$$\mathbf{x}^{(k)} = f^{(k)}(\mathbf{x}^{(k)}) = \sum_{i=0}^n \theta_i^{(k)} f^{(k)}\left(\mathbf{v}_i^{(k)}\right) = \sum_{i=0}^n \theta_i^{(k)} \mathbf{y}_i^{(k)}, \;\; \text{where } \mathbf{y}_i^{(k)} \in f^{(k)}\left(\mathbf{v}_i^{(k)}\right).$$

By compactness of K, the sequences $\left\{\mathbf{x}^{(k)}\right\}_{k=1}^{\infty}$, $\left\{\theta_i^{(k)}\right\}_{k=1}^{\infty}$, and $\left\{\mathbf{y}_i^{(k)}\right\}_{k=1}^{\infty}$, $i = 0, 1, \ldots, n$ converge to $\mathbf{x}^*, \theta_i$ and $\mathbf{y}_i$, respectively, as $k \to \infty$. Then we have $\mathbf{x}^* = \sum_{i=0}^{n} \theta_i \mathbf{y}_i$.

Note that as $k \to \infty$ the diameter of the cell that contains $\mathbf{x}^*$ goes to zero, hence $\mathbf{v}_i^{(k)} \to \mathbf{x}^*$. Thus, by upper semi-continuity of f, from the conditions $\mathbf{v}_i^{(k)} \to \mathbf{x}^*, \mathbf{y}_i^{(k)} \in f(\mathbf{v}_i^{(k)})$ and $\mathbf{y}^{(k)} \to \mathbf{y}_i$ we get that $\mathbf{y}_i \in f(\mathbf{x}^*)$ for all $i = 0, 1, \ldots, n$. Finally, since $f(\mathbf{x}^*)$ is a convex set and $\mathbf{x}^* = \sum_{i=0}^{n} \theta_i \mathbf{y}_i$, we get $\mathbf{x}^* \in f(\mathbf{x}^*)$. □

References

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